left-most:

MAT 127: Calculus C, L03

MAT 127

01/25/22

Need to know

Pre-calc: graphs, exp, log

Calculus A: derivatives

Calculus B: integrals

= "inverse derivatives"

Part I: Ordinary Differential Equations (ODES)

= equations involving a function and its derivatives

E.g. $y' = 5y \qquad y = y(t)$

Part II: Infinite Sequences and Series

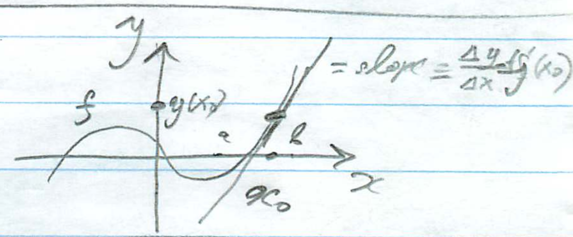
$\cong$ representing functions as "infinite polynomials"

E.g. $\dfrac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$

Derivatives

(1) **Meaning:** f = function of x

- f' = —"—, = rate of change of $f(x)$
- $f'(x_0)$ = slope of tangent line to the graph of $y = f(x)$ at $(x_0, f(x_0))$



- $y'(x) > 0$ on $(a,b) \Leftrightarrow y(x)$ increasing on (a,b)

< 0 decreasing

$= 0$ constant

(2) Derivatives of some functions:

$f(x)$:	x^a	e^x	$\ln x$	$\sin x$	$\cos x$
$f'(x)$:	ax^{a-1}	e^x	$\frac{1}{x}$	$\cos x$	$-\sin x$

$\therefore$ if $f(x) = e^x$, $f' = f$

$\Leftrightarrow g(x) = e^x$ solves diff. equation $y' = y$

(3) Chain rule:

$h(x) = f(g(x)) \Longrightarrow h'(x) = f'(g(x)) \cdot g'(x)$

E.g. $(e^{rx})' = e^{rx} \cdot r$

$f(y) = e^y, \quad y = g(x) = rx$

(4) Product & quotient rules:

$h(x) = f(x) \cdot g(x) \Longrightarrow h'(x) = f'(x)g(x) + f(x)g'(x)$

$h(x) = \dfrac{f(x)}{g(x)} \Rightarrow h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

H/W: (2), (3) + Product rule $\Rightarrow$ Quotient rule

(2), (3), (4) $\Rightarrow$ can compute derivatives of lots of functions

Lots of examples in Chapters 2, 3

Why differential equations (DEs)?

Lots of things can be modelled and studied using diff. equations in science, engineering, economics, ...

Q: if $y=y(t)$ solves $y'=ry$ with $r>0$, what can you say about y?

- if $y(t_0)>0$, $y'(t_0)>0 \Rightarrow y(t)$ increases $\Rightarrow y'(t)$ also increases

$\therefore$ $y(t)$ increases faster and faster

Solution of $y'=ry$

$y(t)=Ce^{rt}$, where C is any fixed constant

To check: Plug in back into DE equation

$y(t)=Ce^{rt} \Rightarrow y'(t)=C\cdot e^{rt}\cdot r \Rightarrow y'(t)\overset{?}{=}ry(t)$

$\therefore$ $y'=ry$ ✓ as in differential equation

Example 2: Logistic Growth Model

limited resources

t= time, $y=y(t)$ population at time t

(rate of change of $y(t)$) is roughly proportional to $y(t)$ when $y(t)$ is very small

$y(t)$ decreases if $y(t)>K$ "carrying capacity"

Example 1: Exponential Growth Model

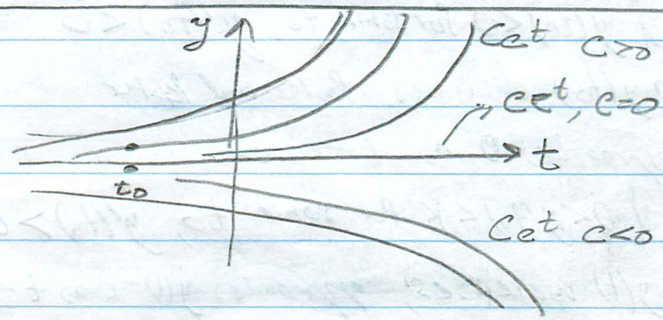
t = time, independent variable

$y(t)$ = (a) population at time t

(b) deposit/debt balance at time t

(rate of change of y at time t) is proportional to y

$y'=ry$ for some $r>0$



if $y(t_0)<0$, $y'(t_0)<0 \Rightarrow y(t)$ decreases, faster & faster

$y(t)=Ce^{rt}$ is the general solution of $y'=ry$

$\Leftrightarrow$ every solution has this form for some C

$y(t)=2e^{rt}$ is a solution

solves $y'=ry$

$\boxed{y'=ry\left(1-\frac{y}{K}\right)}$, $r, K>0$ constants

$y'(t)\approx ry(t)$ if y is small $y \ll K$

$y'(t)<0$ if $y(t)>K$

Q: What do solutions $y(t)$ of this equation look like?

over $\Rightarrow$

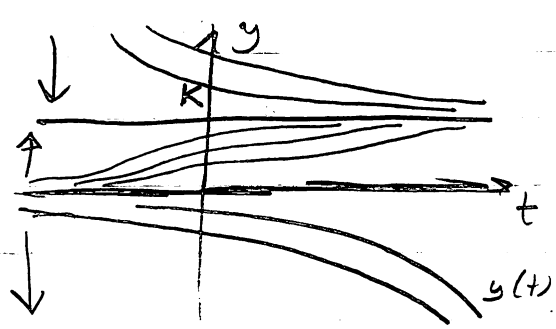
(1) $y(t)=0 \;\; \forall t$ is a solution:

$y'(t)=0 \Rightarrow y' = ry\left(1-\frac{y}{K}\right)$

(2) $y(t)=K \;\; \forall t$ is a solution

$y'(t)=0 \Rightarrow y' = ry\left(1-\frac{y}{K}\right)$

These are constant or <u>equilibrium</u> solutions



(3) if $y(t_0)<0$ for some t_0, $y'(t_0)<0$

$\Rightarrow y(t_0)$ decreases, faster and faster

approaches 0 as $t \to -\infty$

(4) if $0<y(t_0)<K$ for some t_0, $y'(t_0)>0$

$\Rightarrow y(t)$ increases, approaches $y(t)=K$ as $t\to\infty$

0 as $t \to -\infty$

Skip to bottom of p1 of next set
or top of p2