MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 9 (20pts)

WebAssign Problem 1 (5pts)

Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$ *converges or diverges.*

Since the sequence

$$(-1)^n\frac{3n-1}{2n+1}=(-1)^n\frac{(3n-1)/n}{(2n+1)/n}=(-1)^n\frac{3-1/n}{2+1/n}\longrightarrow(-1)^n\frac{3-1/\infty}{2+1/\infty}=(-1)^n\cdot\frac{3}{2}$$

keeps on jumping from near $3/2$ to near $-3/2$ as n approaches ∞, the sequence $(-1)^n\frac{3n-1}{2n+1}$ does not converge to zero. Thus, the series $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$ $\boxed{\text{diverges}}$ by the Test for Divergence.

WebAssign Problem 2 (5pts)

Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n5^n}$ *converges. How many terms of the series are needed to approximate the sum with error less than .0001?*

This series is (strictly) alternating, since the odd terms are negative and the even terms are positive. Furthermore, $1/(n5^n)\longrightarrow 0$ as $n\longrightarrow 0$ and $1/((n+1)5^{n+1})<1/(n5^n)$ for all n. Thus, the series converges by the Alternating Series Test.

We need to find m so that

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n5^n}-\sum_{n=1}^{n=m}\frac{(-1)^n}{n5^n}\right|\le 10^{-4}\,.$$

By the previous paragraph, we can use the *Alternating Series Estimation Theorem*, according to which

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n5^n}-\sum_{n=1}^{n=m}\frac{(-1)^n}{n5^n}\right|\le\frac{1}{(m+1)5^{m+1}}.$$

So, we need m so that $1/(m+1)5^{m+1}\le 1/10^4$, or $(m+1)5^{m+1}>10^4$. The smallest such number m is $\boxed{4}$ (for $m=3$, we get only $4\cdot 5^4=2500$).

Remark: Since the series involves 5^n, we can also use the Ratio Test to show convergence:

$$\frac{|a_{n+1}|}{|a_n|}=\frac{1/((n+1)5^{n+1})}{1/(n5^n)}=\frac{n}{n+1}\cdot\frac{5^n}{5^{n+1}}=\frac{n/n}{(n+1)/n}\cdot\frac{5^n}{5^n\cdot 5^1}=\frac{1}{1+1/n}\cdot\frac{1}{5}\longrightarrow\frac{1}{1+1/\infty}\cdot\frac{1}{5}=\frac{1}{5}\,;$$

since $1/5<1$, the series converges. However, this would not justify the use of the *Alternating Series Estimation Theorem* to answer the second question.

Problem IX.1 (5pts)

Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$ *converges or diverges.*

This series is (strictly) alternating, since the odd terms are negative and the even terms are positive. Furthermore,

$$\frac{n}{\sqrt{n^3+2}}=\frac{n/n}{\sqrt{n^3+2}/n}=\frac{1}{\sqrt{n^3+2}/\sqrt{n^2}}=\frac{1}{\sqrt{(n^3+2)/n^2}}=\frac{1}{\sqrt{n+2/n^2}}\longrightarrow\frac{1}{\sqrt{\infty+0}}=0.$$

Since $((n+1)+2/(n+1)^2)-(n+2/n^2)\geq 1-2/n^2$,

$$\frac{n+1}{\sqrt{(n+1)^3+2}}=\frac{1}{\sqrt{(n+1)+2/(n+1)^2}}<\frac{1}{\sqrt{n+2/n^2}}=\frac{n}{\sqrt{n^3+2}}\qquad\text{if } n\geq 2.$$

Thus, the series $\boxed{\text{converges}}$ by the Alternating Series Test.

Problem IX.2 (5pts)

Approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}$ *correct to four decimal places.*

First find m so that

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}-\sum_{n=1}^{n=m}\frac{(-1)^n}{3^n n!}\right|\leq\frac{1}{2}\cdot 10^{-4}.$$

This series is alternating since the odd terms are negative and the even terms are positive. Further,

$$\lim_{n\longrightarrow\infty}\frac{1}{3^n n!}=0\qquad\text{and}\qquad\frac{1}{3^{n+1}(n+1)!}<\frac{1}{3^n n!}.$$

Thus, we can use the *Alternating Series Estimation Theorem*, according to which

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}-\sum_{n=1}^{n=m}\frac{(-1)^n}{3^n n!}\right|\leq\left|a_{m+1}\right|=\frac{1}{3^{m+1}(m+1)!}.$$

So we need to find m such that $1/3^{m+1}(m+1)!\leq 1/(2\cdot 10^4)$ or $3^{m+1}(m+1)!\geq 2\cdot 10^4$; the smallest m that works is $m=4$ (compute the first four values of $3^{m+1}(m+1)!$). So the required estimate is

$$\sum_{n=1}^{n=4}\frac{(-1)^n}{3^n n!}=\frac{-1}{3\cdot 1!}+\frac{1}{3^2\cdot 2!}+\frac{-1}{3^3\cdot 3!}+\frac{1}{3^4\cdot 4!}$$
$$=\frac{-27\cdot 24+9\cdot 12-3\cdot 4+1}{81\cdot 24}=\frac{-648+108-12+1}{1944}=\boxed{-\frac{551}{1944}}$$

This is an *over*-estimate, since the last term used is positive.

Remark: As we'll see later, the infinite sum is $e^{-1/3}-1\approx -.28347$; so our estimate $-551/1944\approx -.28344$ is indeed within $5\cdot 10^{-5}$ and is an over-estimate.