MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 8 (85pts)

WebAssign Problem 1 (8pts)

Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ *correct to three decimal places.*

We estimate this infinite sum by the finite sum $\sum_{n=1}^{n=m} n^{-5}$ with the smallest possible m such that

$$\sum_{n=1}^{\infty} n^{-5} - \sum_{n=1}^{n=m} n^{-5} = \sum_{n=m+1}^{\infty} n^{-5} \le \frac{1}{2} \cdot \frac{1}{1000}\,.$$

Since $f(x) = x^{-5}$ is a positive, decreasing, and continuous function for $x \ge 1$, by the *Remainder Estimate for the Integral Test Theorem*

$$\int_{m+1}^{\infty} x^{-5}\mathrm{d}x < \sum_{n=m+1}^{\infty} n^{-5} < \int_{m}^{\infty} x^{-5}\mathrm{d}x\,.$$

Since

$$\int_{m}^{\infty} x^{-5}\mathrm{d}x = \frac{1}{-4}x^{-4}\Big|_{m}^{\infty} = -\frac{1}{4}\big(\infty^{-4} - m^{-4}\big) = \frac{1}{4}m^{-4},$$

it follows that

$$\frac{1}{4}(m+1)^{-4} < \sum_{n=m+1}^{\infty} n^{-5} < \frac{1}{4}m^{-4}\,.$$

So we need to find the smallest integer m so that

$$\frac{1}{4m^4} \le \frac{1}{2000} \qquad \Longleftrightarrow \qquad m^4 \ge 500;$$

the smallest integer that does this is $m=5$ (since $4^4 = 256$, while $5^4 = 625$). The estimate is then

$$\sum_{n=1}^{m=5} n^{-5} = \frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} = \frac{60^5 + 30^5 + 20^5 + 15^5 + 12^5}{60^5} = \boxed{\frac{806108207}{777600000} \approx 1.037}$$

WebAssign Problems 2-4 (9+4+3pts)

Determine whether each of the following series converges or diverges:

$$(2)\ \sum_{n=1}^{\infty} n\mathrm{e}^{-n}\,, \qquad (3)\ \sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}\,, \qquad (4)\ \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right).$$

(2) The quickest way here is to use the *Ratio* or *Root Test* (because of e^{-n}):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)\mathrm{e}^{-(n+1)}}{n\mathrm{e}^{-n}} = \left(1+\frac{1}{n}\right)\cdot\frac{\mathrm{e}^{-n}\mathrm{e}^{-1}}{\mathrm{e}^{-n}} = \left(1+\frac{1}{n}\right)\mathrm{e}^{-1} \longrightarrow \left(1+\frac{1}{\infty}\right)\mathrm{e}^{-1} = \mathrm{e}^{-1}\,,$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{n\mathrm{e}^{-n}} = \sqrt[n]{n}\cdot\sqrt[n]{\mathrm{e}^{-n}} = \sqrt[n]{n}\cdot\mathrm{e}^{-1} \longrightarrow 1\cdot\mathrm{e}^{-1} = \mathrm{e}^{-1}\,.$$

Since $\mathrm{e}^{-1} = 1/\mathrm{e} < 1$, the series $\boxed{\text{converges}}$

Alternatively, $0 < \mathrm{e}^{-n/2}$, $\sum_{n=1}^{\infty} \mathrm{e}^{-n/2}$ converges being geometric series with $r = 1/\sqrt{\mathrm{e}} < 1$, and

$$\lim_{n \longrightarrow \infty} \frac{n\mathrm{e}^{-n}}{\mathrm{e}^{-n/2}} = \lim_{n \longrightarrow \infty} n\mathrm{e}^{-n/2} = 0,$$

since the exponential dominates. Thus, $\sum_{n=1}^{\infty} n\mathrm{e}^{-n}$ $\boxed{\text{converges}}$ by the *Limit Comparison Test*.

The *Comparison Test* can be used as well. If $f(x) = x\mathrm{e}^{-x/2}$,

$$f'(x) = x'\mathrm{e}^{-x/2} + x\big(\mathrm{e}^{-x/2}\big)' = \mathrm{e}^{-x/2} + x\mathrm{e}^{-x/2} \cdot (-1/2) = \frac{1}{2}\mathrm{e}^{-x/2}(2-x).$$

So $f(x) \le f(2) = 2\mathrm{e}^{-2/2} < 1$ for $x \ge 2$ and thus $n\mathrm{e}^{-n} \le \mathrm{e}^{-n/2}$ for all n. Since $n\mathrm{e}^{-n} \ge 0$ and $\sum_{n=1}^{\infty} \mathrm{e}^{-n/2}$ converges being geometric series with $r = 1/\sqrt{\mathrm{e}} < 1$, $\sum_{n=1}^{\infty} n\mathrm{e}^{-n}$ also $\boxed{\text{converges}}$

Finally, the *Integral Test* can also be used. The function $f(x) = x\mathrm{e}^{-x}$ is positive and continuous for $x \ge 1$. Since

$$f'(x) = x'\mathrm{e}^{-x} + x\big(\mathrm{e}^{-x}\big)' = \mathrm{e}^{-x} + x\mathrm{e}^{-x} \cdot (-1) = \mathrm{e}^{-x}(1-x),$$

$f(x)$ is decreasing for $x \ge 1$. So the sum converges if and only if $\int_1^{\infty} x\mathrm{e}^{-x}\mathrm{d}x$ does. Integration by parts gives

$$\begin{aligned}
\int_1^{\infty} x\mathrm{e}^{-x}\mathrm{d}x &= -\int_1^{\infty} x\mathrm{d}\mathrm{e}^{-x} = -\left(x\mathrm{e}^{-x}\Big|_1^{\infty} - \int_1^{\infty} \mathrm{e}^{-x}\mathrm{d}x \right) = -\lim_{x \longrightarrow \infty} \left(x\mathrm{e}^{-x} - 1\mathrm{e}^{-1} + \mathrm{e}^{-x} \right)\Big|_1^{\infty} \\
&= -\big(0 - \mathrm{e}^{-1} + \mathrm{e}^{-\infty} - \mathrm{e}^{-1}\big) = 2\mathrm{e}^{-1}.
\end{aligned}$$

Since the integral is finite, $\sum_{n=1}^{\infty} n\mathrm{e}^{-n}$ $\boxed{\text{converges}}$

(3) The terms in this series look like $n^2/n^4 = 1/n^2$. So we limit-compare it to $\sum 1/n^2$; this is a p-series with $p = 2 > 1$ and so converges. This limit-comparison can be made, since both series have positive terms for $n \ge 2$ and

$$\frac{(n^2-1)/(3n^4+1)}{1/n^2} = \frac{(n^2-1)n^2}{3n^4+1} = \frac{(n^2-1)n^2/n^4}{(3n^4+1)/n^4} = \frac{1 - 1/n^2}{3 + 1/n^4} \longrightarrow \frac{1 - 1/\infty}{3 + 1/\infty} = \frac{1}{3}.$$

Since the p-series converges, our series $\boxed{\text{converges}}$ as well.

We can also *compare* (as opposed to *limit-compare*) to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Both series have positive terms for $n \ge 2$ and

$$\frac{n^2-1}{3n^4+1} \le \frac{n^2}{3n^4+1} \le \frac{n^2}{3n^4} = \frac{1}{3n^2}.$$

Since our series is "smaller" than a convergent series, it also $\boxed{\text{converges}}$

(4) The terms in this series look like $1/n$ as $n \longrightarrow \infty$. So we limit-compare it to $\sum 1/n$; this is a p-series with $p = 1 \le 1$ and so diverges. This limit-comparison can be made, since both series have positive terms and

$$\lim_{n \longrightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1.$$

Since the p-series diverges, our series $\boxed{\text{diverges}}$ as well.

We can**not** *compare* (as opposed to *limit-compare*) to the divergent series $\sum 1/n$, as $\sin(1/n) < 1/n$ (a "larger" series being divergent says nothing about the smaller series). We could compare to the divergent series $\sum 1/2n$, since $\sin(x) > x/2$ if $x > 0$ is very small (because $\sin x/x \longrightarrow 1$ as $x \longrightarrow 0$). As both series have positive terms, we could again conclude that our series $\boxed{\text{diverges}}$ (being "larger" than a divergent series).

WebAssign Problems 5,6 (3+3pts)

Determine whether the following series are absolutely convergent:

$$(5) \quad \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^k \qquad (6) \quad \sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}.$$

(5) We need to see if the series $\displaystyle\sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^k$ converges. Since there is $(2/3)^k$, try the *Ratio* or *Root Test:*

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(k+1)(2/3)^{k+1}}{k(2/3)^k} = \frac{k+1}{k} \cdot \frac{(2/3)^{k+1}}{(2/3)^k} = \left(1 + \frac{1}{k}\right) \cdot \frac{(2/3)^k \cdot (2/3)^1}{(2/3)^k}$$
$$= \left(1 + \frac{1}{k}\right) \cdot (2/3) \longrightarrow \left(1 + \frac{1}{\infty}\right) \cdot \frac{2}{3} = \frac{2}{3},$$
$$\sqrt[k]{|a_k|} = \sqrt[k]{k\left(\frac{2}{3}\right)^k} = \sqrt[k]{k} \cdot \sqrt[k]{\left(\frac{2}{3}\right)^k} = \sqrt[k]{k} \cdot \left(\frac{2}{3}\right) \longrightarrow 1 \cdot \left(\frac{2}{3}\right) = \frac{2}{3}$$

Since $2/3 < 1$, the series $\displaystyle\sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^k$ converges. Thus, the original series $\boxed{\text{does}}$ converge absolutely.

(6) Since $\arctan n > 0$ for $n > 0$, we need to see if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$ converges. Since $0 \le (\arctan n)/n^2 \le (\pi/2)/n^2$ and the series

$$\sum_{n=1}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by the p-Series Test with $p = 2 > 1$, the "smaller" series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$ also converges by the *Comparison Test.* Thus, the original series $\boxed{\text{does}}$ converge absolutely.

Problem VIII.1 (5pts)

Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ *converges.*

Since $b^{\ln n} = (e^{\ln b})^{\ln n} = e^{(\ln b)(\ln n)} = e^{(\ln n)(\ln b)} = (e^{\ln n})^{\ln b} = n^{\ln b}$, by the p-series test the series

$$\sum_{n=1}^{\infty} b^{\ln n} = \sum_{n=1}^{\infty} n^{\ln b} = \sum_{n=1}^{\infty} \frac{1}{n^{-\ln b}}$$

converges if and only if $-\ln b > 1$, so that $\boxed{b < 1/e}$

Problem VIII.2 (2+3pts)

Suppose $\sum a_n$ *and* $\sum b_n$ *are series with positive terms and* $\sum b_n$ *is divergent.*
(a) *If* $a_n > b_n$ *for all n, what can you say about* $\sum a_n$*? Why?*
(b) *If* $a_n < b_n$ *for all n, what can you say about* $\sum a_n$*? Why?*

(a) $\sum a_n$ $\boxed{\text{diverges by the Comparison Test}}$
(b) $\boxed{\text{nothing}}$ If $a_n = b_n/2$, $\sum a_n$ diverges; if $a_n = \min(1/2^n, b_n/2)$, $\sum a_n$ converges by the Comparison Test.

Problem VIII.3 (5+5pts)

If $\sum a_n$ *is a convergent series with positive terms, is it true that the series*

$$\text{(a)} \quad \sum \ln(1{+}a_n)\,, \qquad \text{(b)} \quad \sum \sin(a_n)$$

also converges?

(a) $\boxed{\text{Yes}}$ Since $0 < a_n$, $\sum a_n$ converges, and thus $a_n \longrightarrow 0$ and

$$\lim_{n\longrightarrow\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x\longrightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x\longrightarrow 0} \frac{(\ln(1+x))'}{x'} = \lim_{x\longrightarrow 0} \frac{1/(1+x)}{1} = 1.$$

So $\sum \ln(1{+}a_n)$ converges by the *Limit Comparison Test*.

This also follows from the *Comparison Test*, since $0 < \ln(1{+}a_n) \le a_n$ whenever $0 < a_n$, because $1{+}a_n \le e^{a_n}$.

(b) $\boxed{\text{Yes}}$ Since $0 < a_n$, $\sum a_n$ converges, and thus $a_n \longrightarrow 0$ and

$$\lim_{n\longrightarrow\infty} \frac{\sin a_n}{a_n} = \lim_{x\longrightarrow 0} \frac{\sin x}{x} = 1,$$

$\sum \sin(a_n)$ converges by the *Limit Comparison Test*.

This also follows from the *Comparison Test*, since $0 < \sin a_n \le a_n$ whenever $0 < a_n < \pi$.

Problem VIII.4 (5pts)

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ *converges or diverges.*

The terms in this series look like $1/\sqrt{n^3} = n^{-3/2}$. So we limit-compare it to $\sum n^{-3/2}$; this is a p-series with $p = 3/2 > 1$ and so converges. This limit-comparison can be made, since both series have positive terms and

$$\frac{1/\sqrt{n^3+1}}{n^{-3/2}} = \frac{1}{\sqrt{n^3+1}/\sqrt{n^3}} = \frac{1}{\sqrt{(n^3+1)/n^3}} = \frac{1}{\sqrt{1+1/n^3}} \longrightarrow \frac{1}{\sqrt{1+1/\infty}} = 1\,.$$

Since the p-series converges, our series $\boxed{\text{converges}}$ as well.

We can also *compare* (as opposed to *limit-compare*) to the convergent series $\sum n^{-3/2}$. Both series have positive terms and $1/\sqrt{n^3+1} < 1/\sqrt{n^3}$. Since our series is "smaller" than a convergent series, it also $\boxed{\text{converges}}$

Problem VIII.5 (20pts)

For which of the following series is the Ratio Test inconclusive?

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{1}{n^3}\,, \quad \text{(b)}\ \sum_{n=1}^{\infty} \frac{n}{2^n}\,, \quad \text{(c)}\ \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}\,, \quad \text{(d)}\ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}\,.$$

Compute the limit of the ratio of the absolute values of two consecutive terms:

$$\text{(a)}\quad \frac{|a_{n+1}|}{|a_n|} = \frac{1/(n+1)^3}{1/n^3} = \frac{1}{(n+1)^3/n^3} = \frac{1}{((n+1)/n)^3} = \left(\frac{1}{1+1/n}\right)^3 \longrightarrow \left(\frac{1}{1+1/\infty}\right)^3 = 1;$$

$$\text{(b)}\quad \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{(n+1)}{n} \cdot \frac{2^n}{2^n \cdot 2^1} = \left(1+\frac{1}{n}\right)\frac{1}{2} \longrightarrow \left(1+\frac{1}{\infty}\right)\frac{1}{2} = \frac{1}{2};$$

$$\text{(c)}\quad \frac{|a_{n+1}|}{|a_n|} = \frac{3^{(n+1)-1}/\sqrt{n+1}}{3^{n-1}/\sqrt{n}} = \frac{3^{n-1}\cdot 3^1}{3^{n-1}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = 3\frac{1}{\sqrt{n+1}/\sqrt{n}} = \frac{3}{\sqrt{1+1/n}} \longrightarrow \frac{3}{\sqrt{1+1/\infty}} = 3;$$

$$\begin{aligned}\text{(d)}\quad \frac{|a_{n+1}|}{|a_n|} &= \frac{\sqrt{n+1}/(1+(n+1)^2)}{\sqrt{n}/(1+n^2)} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2} = \sqrt{\frac{n+1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(n+1)^2/n^2} \\ &= \sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+((n+1)/n)^2} \\ &= \sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(1+1/n)^2} \longrightarrow \sqrt{1+\frac{1}{\infty}} \cdot \frac{1/\infty+1}{1/\infty+(1+1/\infty)^2} = 1.\end{aligned}$$

Thus, the Ratio Test is inconclusive in $\boxed{\text{(a),(d)}}$

Remark: This problem illustrates the principle that the Ratio Test is *not* suitable for series that involve only powers of n, and not something with faster growth such as 2^n, $n!$, or n^n. While the Ratio Test says nothing about the series in (a) and (d), both converge: (a) by the p-series test and (d) because it looks like $\sqrt{n}/n^2 = 1/n^{3/2}$ (so by Limit Comparison and p-series). By the Ratio

Test, the series in (b) converges, while the series in (c) diverges. These two examples illustrate the principle that the limit obtained in applying the Ratio Test is not affected by factors of n and is just the absolute value of the common ratio r for a geometric series.

Problem VIII.6 (5+5pts)

Determine whether the following series are absolutely convergent:

$$\text{(a)}\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1} \qquad \text{(b)}\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{(2n-1)!}\,.$$

(a) We need to see if the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$ converges. Since $\sqrt{n}/(n+1)$ looks like $\sqrt{n}/n=1/\sqrt{n}$, try the *Limit Comparison Test* with $b_n=1/\sqrt{n}$:

$$\frac{a_n}{b_n}=\frac{\sqrt{n}/(n+1)}{1/\sqrt{n}}=\frac{n}{n+1}=\frac{n/n}{(n+1)/n}=\frac{1}{1+1/n}\longrightarrow\frac{1}{1+1/\infty}=1.$$

Since the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges by the p-Series Test with $p=1/2\le 1$ and both series have positive terms, the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$ also diverges. Thus, the original series $\boxed{\text{does not}}$ converge absolutely.

(b) We need to see if the series $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{(2n-1)!}$ converges. Since there are factorials involved, try the *Ratio Test*:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|}&=\frac{1\cdot 3\cdot\ldots\cdot(2n-1)(2(n+1)-1)/(2(n+1)-1)!}{1\cdot 3\cdot\ldots\cdot(2n-1)/(2n-1)!}\\&=\frac{1\cdot 3\cdot\ldots\cdot(2n-1)(2n+1)}{1\cdot 3\cdot\ldots\cdot(2n-1)}\cdot\frac{(2n-1)!}{(2n+1)!}=(2n+1)\frac{(2n-1)!}{(2n-1)!2n(2n+1)}=\frac{2n+1}{2n(2n+1)}=\frac{1}{2n}\longrightarrow 0\,.\end{aligned}$$

So the series $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{(2n-1)!}$ converges. Thus, the original series $\boxed{\text{does}}$ converge absolutely.

Note: Since the numerator of the nth summand is the product of the odd integers between 1 and $2n-1$ and the denominator is the product of all integers between 1 and $2n-1$, the nth summand is the reciprocal of the product of all even integers between 1 and $2n-1$ (if $n=1$, this product is defined to be 1). Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{(2n-1)!}&=\sum_{n=1}^{\infty}\frac{1}{2\cdot 4\cdot\ldots\cdot(2n-2)}\\&=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}\cdot 1\cdot 2\cdot\ldots\cdot(n-1)}=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}(n-1)!}=\sum_{n=0}^{\infty}\frac{1}{2^n n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\right)^n.\end{aligned}$$

We will see later that this sum is $\mathrm{e}^{1/2}$.