MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 7 (70pts)

Webassign Problem 1 (2pts)

Let $a_n = 2n/(3n+1)$.
(a) *Determine whether* $\{a_n\}$ *converges.*
(b) *Determine whether* $\sum_{n=1}^{\infty} a_n$ *converges.*

Since $a_n = 2/(3+1/n)$, the sequence $\{a_n\}$ $\boxed{\text{converges}}$ to 2/3. Since it does not converge to 0, the corresponding series $\boxed{\text{diverges}}$ by the Test for Divergence of Series.

WebAssign Problem 2 (2pts)

Determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+1)}{(k+3)^2}$ *converges and find its sum if converges.*

The series $\boxed{\text{diverges}}$ because

$$\lim_{k\longrightarrow\infty} \frac{k(k+1)}{(k+3)^2} = \lim_{k\longrightarrow\infty} \frac{1+1/k}{(k+3)^2/k^2} = \lim_{k\longrightarrow\infty} \frac{1+1/k}{(1+3/k)^2} = 1\,.$$

Since the terms in the series do not converge to 0, the series diverges by the Test for Divergence.

WebAssign Problem 3 (5pts)

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ *converges and find its sum by expressing* s_n *as a telescoping sum if converges.*

$$\frac{1}{n^2+4n+3} = \frac{1}{(n+1)(n+3)} = \frac{1}{(+3)-((+1))}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

Thus, for $n \geq 2$

$$s_n = \sum_{k=1}^{k=n}\left(\frac{1}{k+1} - \frac{1}{k+3}\right) = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\,.$$

The second term in each pair with $k \leq n-2$ gets canceled by the first term in the pair $k+2$; this leaves the first terms in the first two pairs and the second terms in the last two pairs. Since $1/n \longrightarrow 0$ as $n \longrightarrow \infty$, the sequence s_n converges. Thus, the series also $\boxed{\text{converges}}$ and

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\longrightarrow\infty} s_n = \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

WebAssign Problems 4 and 5 (5+3pts)

Express the numbers $0.\overline{73} = 0.737373\ldots$ *and* $6.2\overline{54} = 6.2545454\ldots$ *as a ratios of integers.*

$$.\overline{73} = .73 + \frac{.73}{100} + \frac{.73}{100^2} + \ldots = \frac{73/100}{1-\frac{1}{100}} = \boxed{\frac{73}{99}}$$

$$6.2\overline{54} = 6.2 + .054 + \frac{.054}{100} + \frac{.054}{100^2} + \ldots = \frac{62}{10} + \frac{54/1000}{1-\frac{1}{100}} = \frac{31}{5} + \frac{27}{5\cdot 99} = \frac{31}{5} + \frac{3}{5\cdot 11} = \frac{31\cdot 11+3}{55} = \boxed{\frac{344}{55}}$$

WebAssign Problem 6 (4pts)

Find the values of x for which the series $\displaystyle\sum_{n=0}^{\infty}\frac{(x+3)^n}{2^n}$ *converges. Find the sum for those values of x.*

This is a geometric series with $r = (x+3)/2$. Thus, it converges if

$$\left|\frac{x+3}{2}\right| < 1 \quad \Longleftrightarrow \quad |x+3| < 2 \quad \Longleftrightarrow \quad -2 < x+3 < 2 \quad \Longleftrightarrow \quad \boxed{-5 < x < -1}$$

For these values of x,

$$\sum_{n=0}^{\infty}\frac{(x+3)^n}{2^n} = \frac{1}{1-\frac{x+3}{2}} = \frac{2}{2-(x+3)} = \frac{2}{-x-1} = \boxed{-\frac{2}{1+x}}$$

WebAssign Problem 7 (5pts)

If the n-th partial of a series $\displaystyle\sum_{n=1}^{\infty} a_n$ *is* $s_n = \dfrac{n-1}{n+1}$, *find* a_n *and* $\displaystyle\sum_{n=1}^{\infty} a_n$.

Since $s_n = \sum_{k=1}^{k=n} a_k$,

$$a_1 = s_1 = 0, \qquad a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{2}{n(n+1)} \quad \text{if } n \geq 2.$$

By definition, $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\longrightarrow\infty} s_n = \lim_{n\longrightarrow\infty}\frac{1-1/n}{1+1/n} = 1.$

WebAssign Problem 8 (4pts)

Use the Integral Test to determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}$ *converges or diverges.*

Since $f(x) = 1/x^5$ is a positive, decreasing, continuous function for $x \geq 1$, the answer is the same as for

$$\int_1^{\infty}\frac{1}{x^5}\mathrm{d}x = \int_1^{\infty} x^{-5}\mathrm{d}x = \frac{1}{-4}x^{-4}\Big|_{x=1}^{\infty} = -\frac{1}{4}\big(0-1\big) = \frac{1}{4}\,.$$

So the integral converges, and thus the sum $\boxed{\text{converges}}$

Problem VII.1 (5pts)

Determine if the series $\sum_{n=1}^{\infty} \ln(n/(n+1))$ *converges and find its sum by expressing* s_n *as a telescoping sum if converges.*

We have to see if the sequence of partial sum

$$s_n = \sum_{k=1}^{k=n} \ln(k/(k+1)) = \sum_{k=1}^{k=n} \big(\ln k - \ln(k+1)\big) = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \ldots + \big(\ln n - \ln(n+1)\big)$$
$$= \ln 1 - \ln(n+1) = -\ln(n+1)$$

converges. Since $\ln(n+1) \longrightarrow \infty$ as $n \longrightarrow \infty$, the sequence $\{s_n\}$ diverges. Thus, the series $\sum_{n=1}^{\infty} \ln(n/(n+1))$ also $\boxed{\text{diverges}}$

Note: carelessly using telescoping cancellation in this case would give that the sum of the series is $\ln 1 = 0$. This cannot possibly be the case because all terms in the series are negative ($\ln x$ is an increasing function for $x > 0$).

Problem VII.2 (10pts)

The Fibonacci numbers are defined in Section 5.1 by $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ *if* $n \geq 2$. *Show that*

$$\text{(a)}\ \frac{1}{f_{n-1}f_{n+1}} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}, \qquad \text{(b)}\ \sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = 1, \qquad \text{(c)}\ \sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = 2.$$

(a; **2pts**) Since $f_{n+1} = f_n + f_{n-1}$,

$$\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_{n-1}f_n f_{n+1}} = \frac{f_n}{f_{n-1}f_n f_{n+1}} = \frac{1}{f_{n-1}f_{n+1}}.$$

(b; **4pts**) By (a),

$$\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} \right)$$
$$= \left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3} \right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4} \right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5} \right) + \ldots = \frac{1}{f_1 f_2} = 1$$

since the second term in each pair gets canceled by the first term in the following pair **and** $1/f_n f_{n+1} \longrightarrow 0$ (thus leaving only the first term in the first pair).

(c; **4pts**) By (a),

$$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}} \right)$$
$$= \left(\frac{1}{f_1} - \frac{1}{f_3} \right) + \left(\frac{1}{f_2} - \frac{1}{f_4} \right) + \left(\frac{1}{f_3} - \frac{1}{f_5} \right) + \ldots = \frac{1}{f_1} + \frac{1}{f_2} = 2$$

since the second term in each pair gets canceled by the first term two pairs later **and** $1/f_n \longrightarrow 0$ (thus leaving only the first terms in the first two pairs).

Problem VII.3 (5pts)

Find the sum of the series $\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2} \right)$

This is the same as

$$\sum_{n=2}^{\infty} \ln \left(\frac{n^2-1}{n^2} \right) = \sum_{n=2}^{\infty} \big(\ln(n-1) + \ln(n+1) - 2 \ln n \big)$$
$$= \big(\ln 1 + \ln 3 - 2 \ln 2 \big) + \big(\ln 2 + \ln 4 - 2 \ln 3 \big) + \big(\ln 3 + \ln 5 - 2 \ln 4 \big) + \ldots$$

So a lot of terms cancel, and the partial sums are given by

$$s_2 = \ln 1 + \ln 3 - 2 \ln 2 = \ln 3 - 2 \ln 2$$
$$n \geq 3: \qquad s_n = \ln 1 - \ln 2 + \ln(n+1) - \ln n = -\ln 2 + \ln \left(\frac{n+1}{n} \right) = -\ln 2 + \ln \left(1 + \frac{1}{n} \right),$$

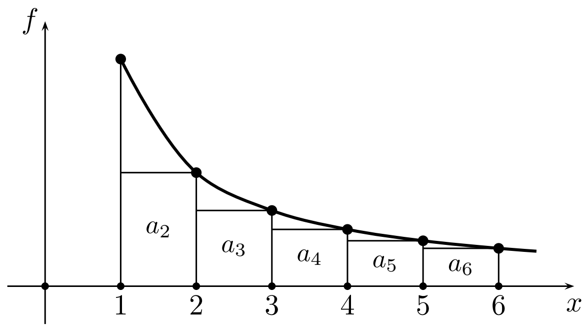
since $-\ln k$ for $k = 3, \ldots, n$ cancels with the middle term in the preceding summand and $-\ln k$ for $k = 2, \ldots, n-1$ cancels with the first term in the following summand. This leaves only the first term in the first summand, the middle term in the last summand, and $-\ln k$ with $k = 2, n$ (the second line in the above displayed expression actually gives the correct expression for $n=2$ as well). By definition, the infinite sum is the limit of the partial sums:

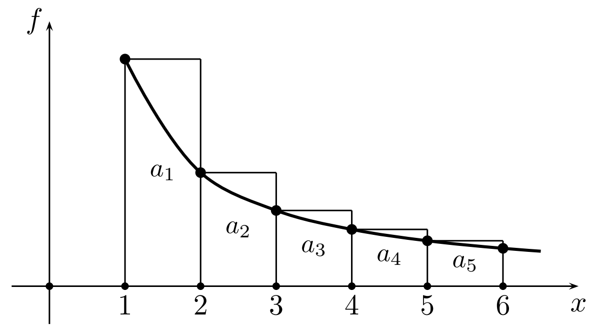
$$\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2} \right) = \lim_{n \longrightarrow \infty} s_n = -\ln 2 + \ln \left(1 + \frac{1}{\infty} \right) = -\ln 2 + \ln 1 = \boxed{-\ln 2}$$

Note: the number of negative and positive terms in s_n is necessarily the same.

Problem VII.4 (5pts)

Let f be a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$. By drawing a picture, rank $\int_1^6 f(x)\mathrm{d}x$, $\sum_{i=1}^{i=5} a_i$, $\sum_{i=2}^{i=6} a_i$ in increasing order.





The 5 rectangles in the first diagram under-estimate the area under the graph of f between $x=1$ and $x=6$; this gives the first inequality in the box below. The 5 rectangles in the second diagram over-estimate the area under the graph of f between $x=1$ and $x=6$; this gives the second inequality in the box below. Both inequalities are equalities if and only if f is constant on the interval [1,6].

$$\boxed{\sum_{i=2}^{i=6} a_i \leq \int_1^6 f(x)\mathrm{d}x \leq \sum_{i=1}^{i=5} a_i}$$

Problem VII.5 (5pts)

For what values of p does the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

If $p \leq 0$, $1/(n(\ln n)^p) \geq 1/n$ whenever $\ln n \geq 1$ (so for $n \geq 3$). Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test and $0 \leq 1/(n(\ln n)^p)$, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges by the *Comparison Test* whenever $p \leq 0$.

Suppose $p > 0$. The function $f(x) = 1/(x(\ln x)^p)$ is then positive, decreasing, and continuous for $x \geq 2$. Thus, the sum converges if and only if

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\mathrm{d}x = \int_{\ln 2}^{\infty} \frac{1}{u^p}\mathrm{d}u$$

does. For $p=1$, we get

$$\int_{\ln 2}^{\infty} \frac{1}{u}\mathrm{d}u = \ln u\Big|_{\ln 2}^{\infty} = \ln\infty - \ln\ln 2 = \infty;$$

so the sum diverges. If $p<1$,

$$\int_{\ln 2}^{\infty} \frac{1}{u^p}\mathrm{d}u = \frac{1}{1-p}u^{1-p}\Big|_{\ln 2}^{\infty} = \frac{1}{1-p}\left(\infty^{1-p} - (\ln 2)^{1-p}\right) = \infty\,,$$

since $1-p>0$; so the sum diverges. Finally, if $p>1$,

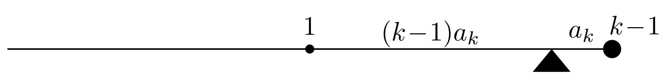
$$\int_{\ln 2}^{\infty} \frac{1}{u^p} \mathrm{d}u = \frac{1}{-p+1} u^{-p+1}\Big|_{\ln 2}^{\infty} = -\frac{1}{p-1}\left(\infty^{-(p-1)} - (\ln 2)^{-(p-1)}\right) = -\frac{1}{p-1}\left(0 - (\ln 2)^{-(p-1)}\right)$$
$$= \frac{1}{p-1}(\ln 2)^{-(p-1)},$$

since $p-1>0$; so the sum converges. Thus, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $\boxed{p>1}$ This is the same answer as in the p-series test.

Problem VII.6 (10pts)

Suppose you have a large supply of bricks, all of the same size, and you stack them at the edge of a table, with each brick extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top brick extends entirely beyond the table. In fact, show that the top brick can extend any distance at all beyond the edge of the table if sufficiently many bricks are used.

We need to stack the bricks so that the center of mass of the top $k-1$ bricks is precisely above the edge of the next brick down or of the table (for the lowest brick); this will keep the stack from collapsing. In this case, the center of mass of the top k brick lies above a point on the lowest of these bricks which is $a_k = 1/2k$ of a brick from the edge of this brick (this center of mass divides the line segment between the center of mass of the lowest of these bricks and the center of mass of the top k–1 bricks in the ratio of (k–1) : 1; see figure below, which depicts the lowest of these bricks).



Thus, the k-th brick from the top extends by $a_k = 1/2k$ beyond the edge of the next brick down. So, if the stack consists of n bricks, the top one will extend by

$$s_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}$$

beyond the edge of the table. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

consists of positive terms and diverges by the p-series test (with $p=1$), its sequence of partial sums $\{s_n\}$, given by the above expression, is increasing to infinity. In particular, for *any* number M we can find n so that $s_n > M$. This says that the far edge of the top brick in a stack of n bricks extends at least distance M beyond the edge of the table (if $M>1$, then the top brick lies entirely beyond the table).

Remark: If you have enough bricks, you can actually stack them so that they extend any distance M beyond the edge of the table *and* you can walk on top of them without having them collapse. *Do you see why?* The argument is very similar...