MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 5 (65pts)

WebAssign Problem 1 (4pts)

Graphs of populations of two species are shown in the first sketch in Figure 1 (the original sketch did not have the line segments labeled P_1 and P_2). Sketch the corresponding phase trajectory and explain your steps.

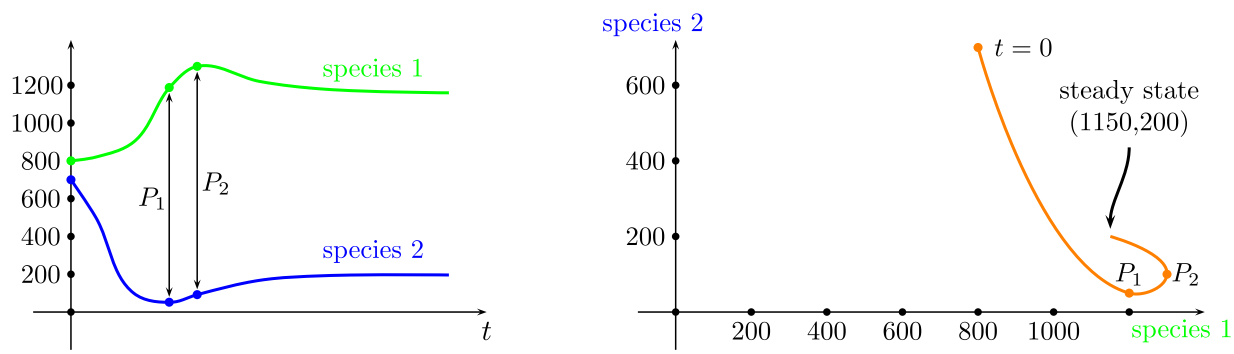

Figure 1: The left diagram shows graphs of population of two species. The right diagram shows the corresponding phase trajectory.

At time $t = 0$, the populations of species 1 and 2 are 800 and 700, respectively, giving the starting point of $(800, 700)$ in the phase plane. The population of the first species then rises (trajectory moves to the right), while the population of the second species falls (trajectory moves down). The latter reaches its minimum at about 50 units, at which point the first population is about 1200 and continues to rise; so the phase trajectory reaches a local minimum at $P_1 \approx (1200, 50)$ and starts moving to the right and up. This continues until the first population reaches about 1300, by which point the second population recovers to about 100; so the phase trajectory passes through $P_2 \approx (1300, 100)$ and then starts going back to the left, but still up (because the second population continues to increase). From then on, the two populations approach a steady state of about (1150,200), with the first population declining toward 1150 and the second rising toward 200 (so the trajectory moves to the left and up). The key points in drawing the phase trajectory are marked on both diagrams in Figure 1; the steady state of (1150,200) should not be marked by a dot, since the curve approaches it as $t \longrightarrow \infty$, but may not actually get there. The scale on the t-axis (shown in the book) plays no role in constructing the phase trajectory.

WebAssign Problem 2 (16pts)

Populations of aphids A and ladybugs L are modeled by the following equations:

$$\begin{cases} \frac{\mathrm{d}A}{\mathrm{d}t} = 2A - .01AL \\ \frac{\mathrm{d}L}{\mathrm{d}t} = -.5L + .0001AL \end{cases} \tag{1}$$

(a) *Find all equilibrium solutions and explain their significance.*

The equilibrium (constant) solutions of (1) are pairs of numbers (A, L) such that

$$\begin{cases} 2A - .01AL = \frac{A}{100}(200 - L) = 0 \\ -.5L + .0001AL = -\frac{L}{10{,}000}(5000 - A) = 0 \end{cases}$$

These pairs satisfy both of the following conditions

$$\begin{cases} A = 0 \text{ or } 200 - L = 0 \\ L = 0 \text{ or } 5000 - A = 0. \end{cases}$$

If we choose the first option on the first line, i.e. $A=0$, then we must choose the first option on the second line, i.e. $L=0$ (because the second option on the second line contradicts our choice from the first line). This gives the equilibrium solution $(A, L) = (0, 0)$, which means there are no aphids or ladybugs ever. On the other hand, if we choose the second option from the first line, i.e. $L=200$, then we must choose the second option on the second line as well, i.e. $A=5000$. So the only other equilibrium solution is $(A, L) = (5000, 200)$; so 5000 aphids are precisely enough to support 200 ladybugs and be contained by them.

Note: A more systematic approach to extracting the equilibrium solutions from the last system of equations above is to write a system of equation for *each* pair consisting of a condition from the first line and a condition from the second line. In this case, we get $2 \cdot 2 = 4$ systems:

$$\begin{cases} A = 0 \\ L = 0 \end{cases} \qquad \begin{cases} A = 0 \\ 5000 - A = 0 \end{cases} \qquad \begin{cases} 200 - L = 0 \\ L = 0 \end{cases} \qquad \begin{cases} 200 - L = 0 \\ 5000 - A = 0 \end{cases}$$

We must then find ALL solutions (A, L) of *each* of these systems. In this case, the second and third systems of equations have no solutions, while the first and the forth give us $(A, L) = (0, 0)$ and $(A, L) = (5000, 200)$, respectively.

(b) *Find an expression for* $\mathrm{d}L/\mathrm{d}A$.

Just divide the second equation in (1) by the first:

$$\frac{\mathrm{d}L}{\mathrm{d}A} = -\frac{L}{100A} \cdot \frac{5000 - A}{200 - L}.$$

(c) *The figure on p546 of Stewart's book shows the direction field for the differential equation in part (b). Use it to sketch a phase plane portrait. What do the phase trajectories have in common?*

The trajectories for the system of the differential equations in (1) travel along the solution curves for the differential equation in (b). These solution curves are everywhere tangent to the little slope lines. In this case, the solution curves are loops going around the equilibrium point $(A, L) = (5000, 200)$, as can be seen from the direction field and is proved in Problem F (the non-trivial part is that these curves are necessarily closed, i.e. circle back to themselves). If $A = 5000$ and $L \in (0, 200)$, i.e. at a point directly below this equilibrium point, $\mathrm{d}A/\mathrm{d}t > 0$ by the first equation in (1), while $\mathrm{d}L/\mathrm{d}t = 0$. Thus, at t increases, the point $(A(t), L(t))$ travels *counter-clockwise* along such a closed curve.

(d) *Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it do describe how both population change.*

This trajectory starts at $(A, L) = (1000, 200)$; this point lies 1/5 of the way from the y-axis to the equilibrium point $(5000, 200)$. By part (c), this trajectory then circles around the point $(5000, 200)$ counter-clockwise. So at first A increases, while L decreases. The trajectory reaches its lowest point when $A = 5000$ (at which point L looks like it might around 100); A then continues to increase, while L starts to increase as well. The trajectory reaches its right-most point when $L = 200$, while A looks like it might be around 15000; A then starts to decrease, while L continues to increases. The trajectory reaches its highest point when $A = 5000$ (at which point L looks like it might around 300); A then continues to decrease, while L starts to decrease as well. The trajectory reaches its left-most point when it returns to the starting point $(A, L) = (1000, 200)$, after which the entire cycle repeats. This is illustrated in the first diagram in Figure 2.

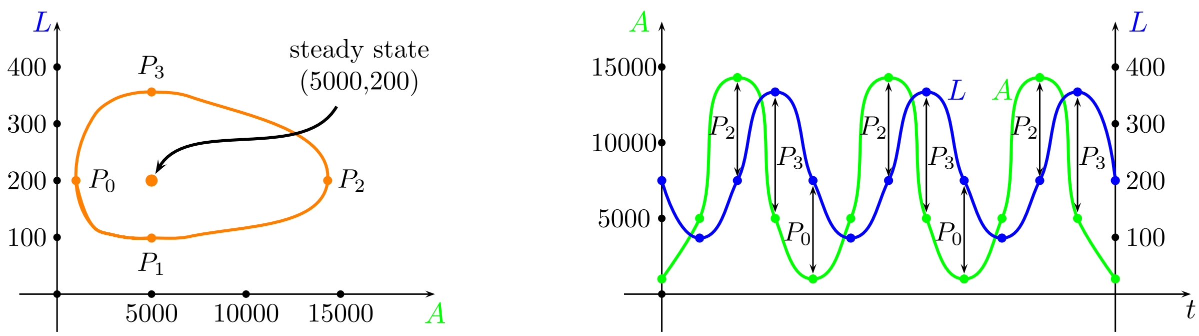

Figure 2: The left diagram shows a phase trajectory. The right diagram shows the corresponding graphs of the functions $A(t)$ and $L(t)$.

Note: we can find an equation for the curve traced by the above trajectory by solving the separable differential in (b) and using the initial condition $(A, L) = (1000, 200)$ to determine the constant. In fact, we can simply use the solution Problem F, with

$$\begin{aligned}
a = 2, \quad b = \frac{1}{100}, \quad c = \frac{1}{2}, \quad d = \frac{1}{10,000} \quad &\Longrightarrow \quad L^2 e^{-L/100} = C A^{-1/2} e^{A/10000} \\
&\Longrightarrow \quad 200^2 e^{-200/100} = C \cdot 1000^{-1/2} e^{1000/10000} \\
&\Longrightarrow \quad C = 4\sqrt{10} \cdot 10^5 \cdot e^{-21/10} \\
&\Longrightarrow \quad L^2 e^{-L/100} = 4 \cdot 10^{11/2} e^{-21/10} A^{-1/2} e^{A/10000}.
\end{aligned}$$

From this, we find that the largest possible value of A is roughly 14302, while the minimum and maximum values of L are roughly 98 and 356, respectively.

(e) *Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?*

This is similar to Problem 7.6 6. First, mark the key points on the phase trajectory in the order they are traversed as t increases (counter-clockwise in the first diagram in Figure 2). These are

- the left-most point $P_0 = (1000, 200)$;

- the lowest point $P_1 \approx (5000, 100)$;
- the right-most point $P_2 \approx (15000, 200)$;
- the highest point $P_3 \approx (5000, 350)$.

Note that both coordinates of P_0 are exact, since this initial point is specified. The first coordinates of P_1 and P_3 are also exact and can be determined from the second equation in (1), since this is where $\mathrm{d}L/\mathrm{d}t = 0$. The second coordinate of P_2 is exact as well and can be determined from the first equation in (1), since this is where $\mathrm{d}A/\mathrm{d}t = 0$. The graphs of $A = A(t)$ and $L = L(t)$ can now be sketched by marking the coordinates of each of the key points of the trajectory on a diagram with horizontal t-axis and two separate vertical axes: A-axis and L-axis. The first coordinates then should be connected by one curve, corresponding to the graph of $A(t)$, while the second coordinates should be connected by another curve, corresponding to the graph of $L(t)$. The two graphs should have no other maxima or minima. While both graphs start at $t=0$, the intermediate t-values cannot be determined from the phase trajectory and so should not be marked on the t-axis. What matters is that the values of A and L for the marked points lie on the same vertical lines; they correspond to the same moments in time, but what these "moments in time" are cannot be determined (except for $t=0$). However, after the A and R return to their starting values, the cycle repeats exactly, taking the same amount of time from the P_0-coordinates to the P_1-coordinates as the first time, and so on.

A rough way in which the two graphs are related is that the L-graph (blue) is a "quarter" of a cycle behind the A-graph (green): the maxima and minima of the former occur a bit after the maxima and minima of the latter.

Note: In order to avoid mixing up the first coordinates (that are used for the A-graph) and the second coordinates (that are used for the L-graphs), either mark them in different colors or with dots and stars, etc. Do not forget to label the axes (t, A, and L in this case) and mark the appropriate scales on the vertical (A and L) axes; these axes should have the same points marked as the corresponding axes in the first diagram in Figure 2. However, the t-axis should carry **no** scale markings (e.g. $t=1$), since the values of t at which the maxima and minima of $A(t)$ and $R(t)$ occur in the second diagram in Figure 2 cannot be determined from the phase trajectory in the first diagram in Figure 2.

Problem V.1 (10pts)

Decide whether each of the following systems of differential equation models two species that compete for the same resources or cooperate for mutual benefit. Explain why each is a reasonable model.

(a) $$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = .12x - .0006x^2 + .00001xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = .08x + .00004xy \end{cases}$$

(b) $$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = .15x - .0002x^2 - .0006xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = .02y - .00008y^2 - .0002xy \end{cases}$$

(a; **5pts**) Since $x'(t)$ increases if y increases and $y'(t)$ increases if x increases, (a) must be a model of $\boxed{\text{cooperation}}$ The number of interactions between the two species is proportional to xy. In $y = 0$, $x' = .12x(1 - x/200)$; this is a logistic growth equation with carrying capacity 200. If $y > 0$, the growth rate of x is increased by an amount proportional to xy and thus to the number of interactions between the species. So the first equation in (a) is a reasonable way to describe the growth of the species x if it benefits from interactions with y, but can also live without y. The growth rate of y is also increased by an amount proportional to the number of interactions between the species. If $x=0$, $y'=0$, while $y'>0$ if $x>0$ even if $y=0$. This is perhaps a reasonable way to model the growth of y if y represents flowering plants (as suggested in the book) which can be pollinated by bees only; or perhaps there is a typo and $.08x$ should be $.08y$.

(b; **5pts**) Since $x'(t)$ decreases if y increases and $y'(t)$ decreases if x increases, (b) must be a model of competition. Since

$$x' = .15x(1 - x/750) \;\text{ if } y = 0 \qquad \text{and} \qquad y' = .02(1 - y/250) \;\text{ if } x = 0$$

in the absence of the species y the growth of x is described by a logistic equation (with carrying capacity of 750 units of x) and in the absence of the species x the growth of y is also described by a logistic equation (with carrying capacity of 250 units of y). Interactions between the two species reduce these growth rates due to competition for the same resources; so (b) is a reasonable model in the given situation.

Problem V.2 (10pts)

A phase trajectory is shown for populations of rabbits (R) and foxes (F) in the left diagram of Figure 3 (the original diagram did not include the points P_1, P_2, P_3 *or the steady state label).*
(a) *Describe how each population changes as the times goes by.*
(b) *Use your description to make a rough sketch of the graphs of* R *and* F *as functions of time.*

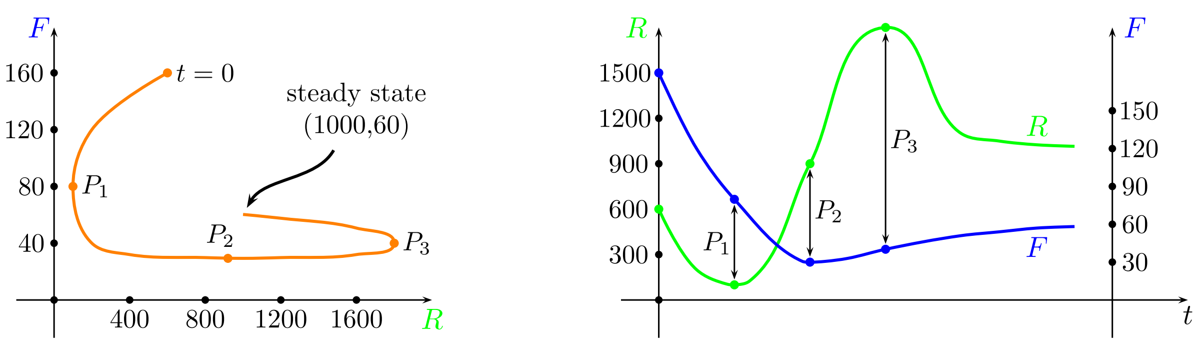

Figure 3: The left diagram shows a phase trajectory. The right diagram shows the corresponding graphs of the functions $R(t)$ and $F(t)$.

At time $t = 0$, there are 160 foxes and about 600 rabbits. The number of both first declines, until the population of foxes reaches 80 and the population of rabbits reaches 100 or so, corresponding to the point P_1 on the phase trajectory. At this point, the population of rabbits starts to increase while the population of foxes continues to decline to about 30, at which moment there are somewhere between 800 and 1000 rabbits (it is hard to tell where the nearly horizontal stretch really reaches its lowest point P_2). The population of foxes then recovers to 40 as the population of rabbits reaches its maximum at about 1800; this corresponds to the point P_3. From then on, the population of rabbits declines to about 1000 and the population of foxes rises to about 60 as $t \longrightarrow \infty$. The key points in drawing the graphs are marked on the second plot in Figure 3; after this the graphs can be sketched. While both graphs start at $t=0$, the intermediate t-values cannot be determined from the phase trajectories and so should not be marked on the t-axis. What matters is the values of R and F for the marked points on the same vertical lines; they correspond to the same moments in time, but what these "moments in time" are cannot be determined (except for $t=0$).

Problem F (25pts)

According to the book, the solutions $(x, y) = (x(t), y(t))$ to the system of differential equations

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + dxy \end{cases} \tag{2}$$

with certain constants $a, b, c, d > 0$ trace simple closed curves (loops) in the xy-plane. Let's see why.

(a; **8pts**) *Divide the second equation in (2) by the first and solve the resulting equation obtaining $y = y(x)$ implicitly; in doing so assume that $x, y > 0$ (so only the first quadrant is considered).*

The division gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-cy + dxy}{ax - bxy} = \frac{y(-c + dx)}{x(a - by)}$$

This equation is separable, so move everything involving y to LHS and everything involving x to RHS and integrate:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(-c + dx)}{x(a - by)} \quad &\iff \quad \frac{a - by}{y}\mathrm{d}y = \frac{(-c + dx)}{x}\mathrm{d}x \\ &\iff \quad \int \big(ay^{-1} - b\big)\mathrm{d}y = \int \big(-cx^{-1} + d\big)\mathrm{d}x \\ &\iff \quad a \ln|y| - by = -c \ln|x| + dx + C. \end{aligned}$$

Using the assumption $x, y > 0$ and exponentiating, we obtain

$$\begin{aligned} \mathrm{e}^{a \ln y - by} = (\mathrm{e}^{\ln y})^a (\mathrm{e}^{-by}) = \mathrm{e}^{-c \ln x + dx + C} = (\mathrm{e}^{\ln x})^{-c} \mathrm{e}^{dx} \mathrm{e}^C \\ \iff \quad \boxed{y^a \mathrm{e}^{-by} = C x^{-c} \mathrm{e}^{dx}} \end{aligned}$$

(b; **10pts**) *Fix the constant C in your general solution (this gives a specific solution of the equation in (a)). Show that the values of $x, y > 0$ that satisfy the equation lie in the interval $[m_C, M_C]$ for some $m_C, M_C > 0$. Furthermore, for each fixed $x > 0$ at most two values of $y > 0$ satisfy the equation; for each fixed $y > 0$ at most two values of $x > 0$ satisfy the equation.*

We need to consider possible pairs (x, y) with $x, y > 0$ that solve the equation

$$G(y) \equiv y^a \mathrm{e}^{-by} = C x^{-c} \mathrm{e}^{dx} \equiv C F(x) \tag{3}$$

for a fixed value of C. Since $G(y) > 0$ if $y > 0$ and $F(x) > 0$ if $x > 0$, the equation (3) has no solutions (x, y) with $x, y > 0$ unless $C > 0$. So we'll assume that $C > 0$ from now on.

Since $a,b>0$, the function $G(y)\longrightarrow 0$ as $y\longrightarrow 0,\infty$; thus, it has a maximum value $\max G$ and is smaller than any $\epsilon>0$ outside of the interval $[m_\epsilon, M_\epsilon]$ for some $m_\epsilon, M_\epsilon>0$. In particular, no value of y satisfies (3) whenever $CF(x)>\max G$. Since $c,d>0$, $F(x)\longrightarrow\infty$ as $x\longrightarrow 0,\infty$; thus, it has a minimum value $\min F>0$ and is larger than $(\max G)/C$ outside of the interval $[m_C, M_C]$ for some $m_C, M_C>0$. Thus, if (x,y) solves (3),

$$C\min F\le CF(x)=G(y)\le \max G. \tag{4}$$

Since this implies that $F(x)\le(\max G)/C$, x lies in the interval $[m_C, M_C]$. Since $G(y)\ge\epsilon=CF(x)$, y lies in some interval $[m'_C, M'_C]$. So, if (x,y) satisfies (3) and thus (4), x and y lie in the interval $[\min(m_C, m'_C), \max(M_C, M'_C)]$; this is the first statement in part (b).

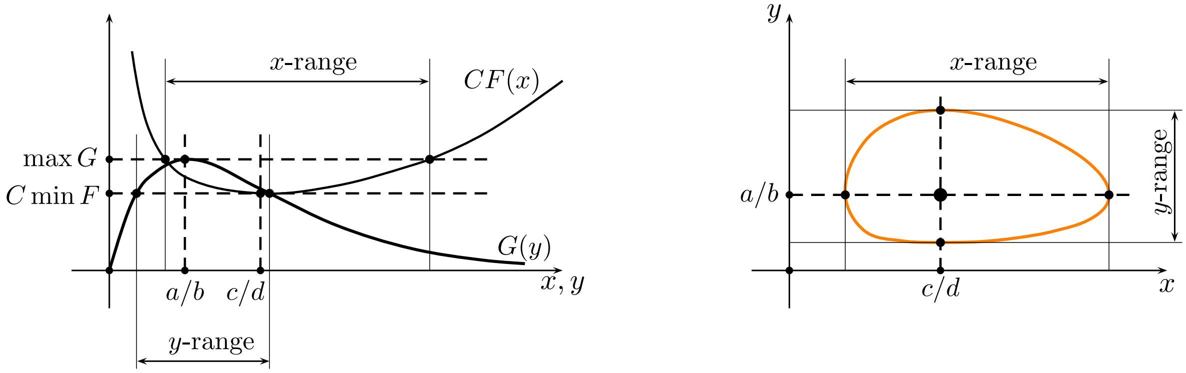

Figure 4: The left diagram shows the graphs of the functions $G(y)$ and $CF(x)$ in (3) with constant $C>0$ fixed. The right diagram shows the curve in the xy-plane described by (3) with the same fixed C. A solution $(x(t),y(t))$ of (2) goes around this curve counter-clockwise as t increases.

The function $G(y)$ is in fact first increasing and then decreasing, i.e. it has only one critical point:

$$G'(y)=(y^a)'\mathrm{e}^{-by}+y^a(\mathrm{e}^{-by})'=ay^{a-1}\mathrm{e}^{-by}+y^a(-b)\mathrm{e}^{-by}=y^{a-1}\mathrm{e}^{-by}(a-by);$$

so $G'(y)=0$ only for $y=b/a$. This implies that there are at most two values of y with the same value of $G(y)$; so for each fixed x, there are at most two values of y that satisfy (3). Similarly, the function $F(x)$ is first decreasing and then increasing, i.e. it has only one critical point:

$$F'(x)=(x^{-c})'\mathrm{e}^{dx}+x^{-c}(\mathrm{e}^{dx})'=-cx^{-c-1}\mathrm{e}^{dx}+x^{-c}d\mathrm{e}^{dx}=x^{-c-1}\mathrm{e}^{dx}(-c+dx);$$

so $F'(x)=0$ only for $x=d/c$. This implies that there are at most two values of x with the same value of $F(x)$; so for each fixed y, there are at most two values of x that satisfy (3). This gives the remaining statement in part (b).

(c; **2pts**) *Assuming $x,y>0$, show that $(x'(t),y'(t))=0$ if and only if $(x(t),y(t))=(c/d,a/b)$.*
By equation (2), $(x'(t),y'(t))=0$ is equivalent to

$$\begin{cases} x(a-by)=0\\ y(-c+dx)=0.\end{cases}$$

Since $x,y>0$, this is equivalent to $(x(t),y(t))=(c/d,a/b)$.

(d; **5pts**) *Show that every phase trajectory of (2) in the first quadrant of the xy-plane other than the equilibrium point $(c/d, a/b)$ repeatedly traces a closed curve enclosing $(c/d, a/b)$ in the counter-clockwise direction.*

For each value of the constant C, your solution in (a) is an equation for the curve traced by a solution $(x, y) = (x(t), y(t))$ to the system (2) in the xy-plane. By (b), this curve is contained in a rectangle, and every vertical and horizontal line intersects the curve at most twice. Thus, either the curve is closed and $(x(t), y(t))$ keeps on moving around it as t increases or $(x'(t), y'(t))$ approaches (0,0) as $t \longrightarrow \infty$ or $(x'(t_0), y'(t_0)) = (0, 0)$ some t_0 (so that the path $(x(t), y(t))$ can reverse direction on the curve). By (c), the last two things can (and do) occur only on the "curve" containing the equilibrium point $(c/d, a/b)$; so each of the non-equilibrium solutions (2) keeps on going around some closed curve, containing the equilibrium point.

If (x, y) is a solution of (3), $CF(x)$ cannot exceed $\max G$; so the possible values of x correspond to the segment of the graph of $CF(x)$ that lies below $\max G$. At the two endpoints of this range of x, $CF(x) = \max G$; so the corresponding value of y in (3) is $y = a/b$, since this is where $G(y)$ reaches $\max G$. Thus, the left-most and right-most points on the curve (3) in the xy-plane have the same y-coordinate a/b. Similarly, if (x, y) is a solution of (3), $G(y)$ cannot be smaller than $C \min F$; so the possible values of y correspond to the segment of the graph of $G(y)$ that lies above $C \min F$. At the two endpoints of this range of y, $G(y) = C \min F$; so the corresponding value of x in (3) is $x = c/d$, since this is where $CF(x)$ reaches $C \min F$. Thus, the lowest and highest points on the curve (3) in the xy-plane have the same x-coordinate c/d. The statements concerning the extremal points of the curve are true for all solution curves of (2) in the phase, xy-plane. The equilibrium point is $(c/d, a/b)$.

For example, in Figures 2 and 3 in 7.6, the minimum R-value and the maximum R-value on any solution curve are both reached when $W = 80$; the minimum W-value and the maximum W-value on any solution curve are both reached when $R = 1000$. The equilibrium point is $(R, W) = (1000, 80)$.

Note: The intersections of the graphs of $CF(x)$ and $G(y)$ in the first sketch in Figure 4 are completely irrelevant and even meaningless, since the horizontal coordinates, x and y, might be measured in completely different physical units (for example, hundreds of rabbits and dozens of wolves). What matters is the part of the graph of $CF(x)$ that lies below the top of the graph of $G(y)$ and the part of the graph of $G(y)$ that lies above the bottom of the graph of $CF(x)$.

Remark: In an actual differential equations course, such as MAT 303/305, the topic of systems of two autonomous equations, such as (2), can easily take up a month. After that, it is possible to draw phase-plane portraits (solution curves in the xy-plane, where $(x, y) = (x(t), y(t))$) such as the right diagrams at the bottom of p11 and p10 in

http://www.math.stonybrook.edu/∼azinger/mat127-spr22/hw5/ODE1sol.pdf
http://www.math.stonybrook.edu/∼azinger/mat127-spr22/hw5/ODE2sol.pdf

in less than half an hour (each of these was one of 10 problems on a 3-hour final exam, but worth 1/6 of the points on the exam).