MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 3 (85pts)

Webassign Problem 1 (10pts)

Find the general solution to the differential equation

$$(x^2+1)y' = xy. \tag{1}$$

This is a separable equation. Write $y' = \mathrm{d}y/\mathrm{d}x$, move everything involving y to LHS and everything involving x to RHS, and integrate:

$$\begin{aligned}
(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} = xy \qquad &\iff \qquad \frac{\mathrm{d}y}{y} = \frac{x}{x^2+1}\mathrm{d}x \\
&\iff \qquad \int \frac{\mathrm{d}y}{y} = \int \frac{x}{x^2+1}\mathrm{d}x \\
&\iff \qquad \ln|y| = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C,
\end{aligned}$$

where C is any constant. Exponentiating both sides, we obtain

$$|y| = \mathrm{e}^{\ln\sqrt{x^2+1}+C} = \mathrm{e}^{\ln\sqrt{x^2+1}} \cdot e^C = A\sqrt{x^2+1} \qquad \iff \qquad y = \pm A\sqrt{x^2+1}; \tag{2}$$

since C is any constant, $A = e^C$ is any *positive* constant. However, we divided both sides of (1) by y. This is 0 if $y = 0$ for all x; this gives rise to the only constant solution of the differential equation, which in turn corresponds to $A = 0$ in (2). So the general solution (i.e. the set of all solutions) of (1) is $\boxed{y(x) = C\sqrt{x^2+1}}$ where C is now any constant.

Note: It is good to check that the function $y = y(x)$ is indeed a solution of (1) by computing y', $(x^2+1)y'$, and xy and comparing the last two.

Webassign Problem 2 (8pts)

Find the solution to the initial-value problem

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \sqrt{Pt}\,, \qquad P(1) = 2. \tag{3}$$

First find the general solution of the differential equation. This is a separable equation, so we can move everything involving P to LHS and everything involving t to RHS, and then integrate:

$$\begin{aligned}
\frac{\mathrm{d}P}{\sqrt{P}} = \sqrt{t}\,\mathrm{d}t \qquad &\iff \qquad \int P^{-1/2}\mathrm{d}P = \int t^{1/2}\,\mathrm{d}t \\
&\iff \qquad 2P^{1/2} = \frac{2}{3}t^{3/2} + C,
\end{aligned} \tag{4}$$

where C is any constant. Dividing by 2 and squaring both sides of (4) gives

$$P = \left(\frac{1}{3}t^{3/2} + C/2\right)^2. \tag{5}$$

Since C is any constant, so is $C/2$; so we can replace $C/2$ above by C. However, we divided both sides of (3) by $\sqrt{P}$. This is 0 if $P = 0$ for all t; this gives rises to the only constant solution of the differential equation. So the general solution (i.e. the set of all solutions) of (3) is

$$P = \left(\frac{1}{3}t^{3/2} + C\right)^2, \qquad P = 0, \tag{6}$$

where C is now any constant.

It remains to determine which of the solutions (6) of the differential equation (3) satisfies the initial condition $P(1) = 2$. The constant solution $P(t) = 0$ for all t does not satisfy this condition. So we need to find C so that the function $P = P(x)$ defined by the first equation in (6) satisfies the initial condition $P(1)=2$ in (3). For this, plug in $t = 1$ and $P = 2$ into the first equation in (6):

$$2 = \left(\frac{1}{3} \cdot 1^{3/2} + C\right)^2 \quad \Longrightarrow \quad \sqrt{2} = \frac{1}{3} + C \quad \Longrightarrow \quad C = \sqrt{2} - \frac{1}{3}.$$

So the solution to the initial-value problem (3) is given by $\boxed{P(t) = \left(\frac{1}{3}t^{3/2} - \frac{1}{3} + \sqrt{2}\right)^2}$

Note 1: It is good to check that the function $P = P(t)$ above actually solves (3). So compute P' to compare it with $\sqrt{Pt}$ and check that $P(1) = 2$ as required by the initial condition in (3). Since the latter is easier, it should probably be done first.

Note 2: Another way to find the solution to (3) is to find C in the last expression in (4) by plugging in the initial condition $(t, P) = (1, 2)$:

$$2 \cdot 2^{1/2} = \frac{2}{3} \cdot 1^{3/2} + C \qquad \Longrightarrow \qquad C = 2\sqrt{2} - \frac{2}{3}.$$

So the solution $P = P(t)$ to (3) satisfies

$$2P^{1/2} = \frac{2}{3}t^{3/2} - \frac{2}{3} + 2\sqrt{2}.$$

Dividing by 2 and squaring both sides, we recover the answer obtained above. Finding the correct constant as early as possible is generally easier, though less systematic than the first approach.

Webassign Problem 3 (10pts)

Find an equation for the curve passing through (0,1) and whose slope at (x, y) is xy.

The slope of the graph of $y = y(x)$ at $(x, y(x))$ is $y'(x)$. So we need to solve the initial-value problem

$$y' = xy, \qquad y(0) = 1. \tag{7}$$

First find the general solution of the differential equation. This is a separable equation, so after writing $y' = \mathrm{d}y/\mathrm{d}x$, we can move everything involving y to LHS and everything involving x to RHS, and then integrate:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = xy \quad \Longleftrightarrow \quad \frac{\mathrm{d}y}{y} = x\mathrm{d}x \quad &\Longleftrightarrow \quad \int \frac{\mathrm{d}y}{y} = \int x\mathrm{d}x \quad \Longleftrightarrow \quad \ln|y| = \frac{1}{2}x^2 + C \\
&\Longleftrightarrow \quad \mathrm{e}^{\ln|y|} = \mathrm{e}^{\frac{1}{2}x^2 + C} = \mathrm{e}^{\frac{1}{2}x^2} \cdot \mathrm{e}^C \\
&\Longleftrightarrow \quad |y| = \mathrm{e}^C \mathrm{e}^{\frac{1}{2}x^2} \quad \Longleftrightarrow \quad y = A\mathrm{e}^{\frac{1}{2}x^2},
\end{aligned} \tag{8}$$

where C is any constant and $A = \pm e^C$ is any nonzero constant. However, we divided both sides of (7) by y. This is 0 if $y = 0$ for all x; this gives rise to the only constant solution of the differential equation, which corresponds to $A=0$. We now need to find A so that the function y in the last equation in (8) satisfies the initial condition $y(0)=1$. For this, plug in $(x,y) = (0,1)$ into the last equation in (8):

$$1 = Ae^{\frac{1}{2}0^2} = A.$$

So $A = 1$, and an equation for the specified curve is $\boxed{y = e^{\frac{1}{2}x^2}}$

Note 1: It is good to check that the function $y = y(x)$ actually solves (7): so compute y' to compare it with xy and check that $y(0) = 1$ as required by the initial condition in (7). Since the latter is easier, it should probably be done first.

Note 2: Another way to find the solution to (7) is to find C in the first expression in (8) containing it by plugging in the initial condition $(x,y) = (0,1)$:

$$\ln|1| = \frac{1}{2}0^2 + C \qquad \Longrightarrow \qquad C = 0.$$

So, the solution $y = y(x)$ to (7) satisfies

$$\ln|y| = \frac{1}{2}x^2 \qquad \Longleftrightarrow \qquad |y| = e^{\frac{1}{2}x^2} \qquad \Longleftrightarrow \qquad y = \pm e^{\frac{1}{2}x^2}.$$

Since $y(0)>0$, this again gives $y(x) = e^{x^2/2}$.

Webassign Problem 4 (11pts)

A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

(a) *How much salt is in the tank after t minutes?*
Let $y(t)$ be the amount of salt in the tank, in kgs, at time t, in minutes. Thus, $y(0)=15$ and

$$y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t),$$

where

$$y'_{\text{in}}(t) = (\text{flow rate of salt})_{\text{in}} = (\text{flow rate of solution})_{\text{in}} \cdot (\text{flow concentration})_{\text{in}} = 10 \cdot 0\% = 0;$$
$$y'_{\text{out}}(t) = (\text{flow rate of salt})_{\text{out}} = (\text{flow rate of solution})_{\text{out}} \cdot (\text{flow concentration})_{\text{out}}.$$

Since the solution in the tank is kept thoroughly mixed, the outgoing flow concentration is the same as the salt concentration in the tank:

$$(\text{flow concentration})_{\text{out}} = \frac{\text{amount of salt in tank}}{\text{volume in tank}} = \frac{y(t)}{1000},$$

since the volume of solution in the tank is kept constant at 1000 gallons. So,

$$y'_{\text{out}}(t) = 10 \cdot \frac{y(t)}{1000} = \frac{y(t)}{100}.$$

It follows that $y(t)$ is the solution to the initial-value problem

$$y'(t) = 0 - \frac{y(t)}{100} = -\frac{1}{100}y(t)\,, \qquad y(0) = 15.$$

Since this is just the exponential decay equation, the solution to this initial-value problem is

$$y(t) = y(0)\mathrm{e}^{-\frac{1}{100}t} = \boxed{15\mathrm{e}^{-t/100}}$$

Alternatively, we can find the general solution of the differential equation and the particular solution satisfying the initial condition. Since the differential equation is separable, writing $y' = \mathrm{d}y/\mathrm{d}t$, moving everything involving y to LHS and everything involving t to the RHS, and integrating, we obtain

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{100}y \quad &\Longleftrightarrow \quad \frac{\mathrm{d}y}{y} = -\frac{1}{100}\mathrm{d}t \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}y}{y} = -\int \frac{1}{100}\mathrm{d}t \quad \Longleftrightarrow \quad \ln|y| = -\frac{t}{100} + C \\ &\Longleftrightarrow \quad \mathrm{e}^{\ln|y|} = \mathrm{e}^{-t/100+C} = \mathrm{e}^{-t/100}\cdot \mathrm{e}^{C} \quad \Longleftrightarrow \quad \ln|y| = \mathrm{e}^{C}\mathrm{e}^{-t/100} \\ &\Longleftrightarrow \quad y = A\mathrm{e}^{-t/100}\,,\end{aligned}$$

where C is any constant and $A = \pm\mathrm{e}^{-t/100}$ is any nonzero constant. However, we divided the equation by y, which is 0 if $y = 0$; this gives us the constant solution $y = 0$ of the differential equation, which corresponds to $A = 0$. In order to find A, plug in the initial condition $(t, y) = (0, 15)$ into the last equation above:

$$15 = A\mathrm{e}^{-0/100} \quad \Longrightarrow \quad A = 15 \quad \Longrightarrow \quad \boxed{y(t) = 15\mathrm{e}^{-t/100}}$$

We could also find the particular solution by plugging in the initial condition $(t, y) = (0, 15)$ into the first equation equation above that contains C:

$$\begin{aligned}\ln|15| = -\frac{0}{100} + C \quad &\Longrightarrow \quad C = \ln 15 \quad \Longrightarrow \quad \ln|y| = -\frac{t}{100} + \ln 15 \\ &\Longrightarrow \quad \mathrm{e}^{\ln|y|} = \mathrm{e}^{-\frac{t}{100}+\ln 15} = \mathrm{e}^{-\frac{t}{100}}\cdot \mathrm{e}^{\ln 15} = 15\mathrm{e}^{-\frac{t}{100}} \\ &\Longrightarrow \quad |y| = 15\mathrm{e}^{-\frac{t}{100}} \quad \Longrightarrow \quad y = \pm 15\mathrm{e}^{-\frac{t}{100}}.\end{aligned}$$

Since $y(t)$ cannot be negative, the sign above must be $+$ and we recover the same formula for $y(t)$.

Note: As a reality check, note that $y(t)$ approaches 0 as $t \longrightarrow \infty$, as expected because the salt gets washed out with the pure water being poured into the tank.

(b) *How much salt is in the tank after 20 minutes?*
Since in the above formulas for $y(t)$ the time t is measured in minutes, we can simply plug in $t = 20$:

$$y(20) = 15\mathrm{e}^{-20/100} = \boxed{15\mathrm{e}^{-1/5} \approx 12.28 \text{ kg}}$$

Note the units.

Webassign Problem 5 (11pts)

A tank contains 1000L of pure water. Brine that contains .05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Brine that contains .04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate 15 L/min. How much salt is in the tank after (a) *t minutes;* (b) *one hour.*

Let $y(t)$ be the amount of salt in the tank, in kilograms, at time t, in minutes. Thus, $y(0) = 0$. Furthermore, $y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t)$, where

$$y'_{\text{in}}(t) = (\text{flow rate of salt})_{\text{in}}, \qquad y'_{\text{out}}(t) = (\text{flow rate of salt})_{\text{out}}.$$

Since in this case two different salt solutions are entering the tank,

$$\begin{aligned} y'_{\text{in}}(t) &= (\text{flow rate of brine})_{\text{in};1} \cdot (\text{flow concentration})_{\text{in};1} \\ &\quad + (\text{flow rate of brine})_{\text{in};2} \cdot (\text{flow concentration})_{\text{in};2} = 5 \cdot .05 + 10 \cdot .04 = \frac{25+40}{100} = \frac{13}{20}. \end{aligned}$$

Similarly to the previous problem,

$$\begin{aligned} y'_{\text{out}}(t) &= (\text{flow rate of salt})_{\text{out}} = (\text{flow rate of solution})_{\text{out}} \cdot (\text{flow concentration})_{\text{out}} \\ &= (5+10) \cdot (\text{flow concentration})_{\text{tank}} = 15 \cdot \frac{\text{amount of salt in tank}}{\text{volume in tank}} = 15 \cdot \frac{y(t)}{1000} = \frac{3}{200} y(t), \end{aligned}$$

since the volume in the tank is constant at 1000 liters. It follows that $y(t)$ is the solution to the initial-value problem

$$y'(t) = \frac{13}{20} - \frac{3}{200} y(t) = \frac{130 - 3y(t)}{200}, \qquad y(0) = 0.$$

First find the general solution to the differential equation. Since it is separable, writing $y' = \mathrm{d}y/\mathrm{d}t$, moving everything involving y to LHS and everything involving t to the RHS, and integrating, we obtain

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{130-3y}{200} \quad &\Longleftrightarrow \quad \frac{\mathrm{d}y}{130-3y} = \frac{\mathrm{d}t}{200} \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}y}{130-3y} = \int \frac{\mathrm{d}t}{200} \\ &\Longleftrightarrow \quad -\frac{1}{3} \ln|130-3y| = \frac{t}{200} + C \quad \Longleftrightarrow \quad \ln|130-3y| = -\frac{3t}{200} + C \\ &\Longleftrightarrow \quad \mathrm{e}^{\ln|130-3y|} = \mathrm{e}^{-3t/200+C} = \mathrm{e}^{C}\mathrm{e}^{-3t/200} \quad \Longleftrightarrow \quad \ln|130-3y| = A\mathrm{e}^{-3t/200} \\ &\Longleftrightarrow \quad 130 - 3y = \pm A\mathrm{e}^{-3t/200} \quad \Longleftrightarrow \quad y(t) = \frac{130}{3} + C\mathrm{e}^{-3t/200}. \end{aligned}$$

Plugging in the initial condition $(t, y) = (0, 20)$, we obtain

$$0 = \frac{130}{3} + C\mathrm{e}^{-3 \cdot 0/200} = \frac{130}{3} + C \qquad \Longleftrightarrow \qquad C = -\frac{130}{3}.$$

So the amount of salt in the tank after t minutes is $\boxed{y(t) = \frac{130}{3}(1 - \mathrm{e}^{-3t/200}) \text{ kg}}$. Thus, the amount of salt in the tank after 1 hour is

$$y(60) = \frac{130}{3}(1 - \mathrm{e}^{-180/200}) = \boxed{\tfrac{130}{3}(1 - \mathrm{e}^{-9/10}) \approx 25.72 \text{ kg}}$$

Note: As a reality check, note that the salt concentration $\rho(t) = y(t)/1000$ approaches the weighted average concentration of the incoming solutions, which is $(13/20)/15 = (130/3)/1000$.

Remark (for Webassign Problems 4,5): On the exams, you will need to leave your answers in an exact form, **as simple as possible**, even if they involve exponentials and logs.

Problem C (35pts)

A ball of mass m is projected vertically upward from the earth's surface with a positive velocity v_0. The forces acting on the ball are the force of gravity and the air resistance; the magnitude of the latter is proportional to the speed (the magnitude of the velocity). So, by Newton's Second Law, the equation of motion is

$$mv' = ma = -mg - pv,$$

where g and p are positive constants.

(a; **8pts**) *Show that the upward velocity of $v=v(t)$, until the ball returns to the ground, is given by*

$$v(t) = \Big(v_0 + \frac{mg}{p}\Big)\mathrm{e}^{-pt/m} - \frac{mg}{p}\,.$$

The upward velocity of $v=v(t)$ is the solution to the initial-value problem

$$mv' = -mg - pv, \qquad v(0) = v_0.$$

The above function satisfies the second condition because

$$v(0) = \Big(v_0 + \frac{mg}{p}\Big)\mathrm{e}^{-p\cdot 0/m} - \frac{mg}{p} = \Big(v_0 + \frac{mg}{p}\Big) - \frac{mg}{p} = v_0\,.$$

It satisfies the differential equation because

$$v'(t) = \Big(v_0 + \frac{mg}{p}\Big)\mathrm{e}^{-pt/m}\cdot\frac{-p}{m} \qquad \Longrightarrow \qquad mv' = -p\Big(v_0 + \frac{mg}{p}\Big)\mathrm{e}^{-pt/m}\,,$$
$$-mg - pv = -p\Big(v_0 + \frac{mg}{p}\Big)\mathrm{e}^{-pt/m}\,;$$

so LHS of the differential equation equals to RHS of the differential equation when we plug in the above function $v=v(t)$.

Alternatively, we can first find the general solution of the differential equation and then the particular solution satisfying the initial condition. This is a separable equation, so after writing $v' = \mathrm{d}v/\mathrm{d}t$, we can move everything involving v to LHS and everything involving t to RHS, and then integrate:

$$\begin{aligned} m\frac{\mathrm{d}v}{\mathrm{d}t} = -(mg+pv) \quad &\Longleftrightarrow \quad \frac{m\,\mathrm{d}v}{mg+pv} = -\mathrm{d}t \quad \Longleftrightarrow \quad m\int\frac{\mathrm{d}v}{mg+pv} = -\int\mathrm{d}t \\ &\Longleftrightarrow \quad \frac{m}{p}\ln|mg+pv| = -t + C \quad \Longleftrightarrow \quad \ln|mg+pv| = -\frac{pt}{m} + \frac{p}{m}C \\ &\Longleftrightarrow \quad \mathrm{e}^{\ln|mg+pv|} = \mathrm{e}^{-\frac{pt}{m}+\frac{p}{m}C} = \mathrm{e}^{-\frac{pt}{m}}\cdot\mathrm{e}^{\frac{p}{m}C} \quad \Longleftrightarrow \quad |mg+pv| = \mathrm{e}^{\frac{p}{m}C}\mathrm{e}^{-\frac{pt}{m}} \\ &\Longleftrightarrow \quad mg+pv = \pm\mathrm{e}^{\frac{p}{m}C}\mathrm{e}^{-\frac{pt}{m}} \quad \Longleftrightarrow \quad pv = -mg + A\mathrm{e}^{-\frac{pt}{m}}\,, \end{aligned}$$

where C is any constant and $A = \pm\mathrm{e}^{pC/m}$ is any nonzero constant. However, we divided both sides of our equation by $mg+pv$. This is 0 if $pv = -mg$ for all t; this gives rise to the only constant solution of the differential equation, which corresponds to $A=0$. We now need to find A so that the function v defined above satisfies the initial condition $v(0)=v_0$. For this, plug in $(t,v) = (0,v_0)$ into the last equation above:

$$pv_0 = -mg + A\mathrm{e}^{-\frac{p\cdot 0}{m}} = -mg + A \quad \Longrightarrow \quad A = pv_0 + mg \quad \Longrightarrow \quad pv = \big(pv_0+mg\big)\mathrm{e}^{-pt/m} - mg\,,$$

as claimed in the statement of the problem.

Alternatively, we could plug in $(t,v)=(0,v_0)$ into the first expression above that contains C:

$$\begin{aligned}
&\frac{m}{p}\ln|mg{+}pv_0| = -0+C \quad\Longrightarrow\quad C=\frac{m}{p}\ln|mg{+}pv_0| \\
&\qquad\Longrightarrow\quad \frac{m}{p}\ln|mg{+}pv| = -t+\frac{m}{p}\ln|mg{+}pv_0| \quad\Longrightarrow\quad \ln|mg{+}pv| = -\frac{mt}{p}+\ln|mg{+}pv_0| \\
&\qquad\Longrightarrow\quad \ln|mg{+}pv| = \mathrm{e}^{-\frac{mt}{p}+\ln|mg+pv_0|} = \mathrm{e}^{-\frac{mt}{p}}\cdot \mathrm{e}^{\ln|mg+pv_0|} = |mg{+}pv_0|\cdot \mathrm{e}^{-\frac{mt}{p}} \\
&\qquad\Longrightarrow\quad mg{+}pv = \pm\big(mg{+}pv_0\big)\mathrm{e}^{-\frac{mt}{p}}\,.
\end{aligned}$$

Since $v(0){=}v_0$, the sign above must be $+$, and we again recover

$$pv(t) = \big(mg{+}pv_0\big)\mathrm{e}^{-\frac{mt}{p}} - mg,$$

as stated in the problem.

(b; **4pts**) *Show that the height $y{=}y(t)$ of the ball, until it hits the ground, is given by*

$$y(t) = \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{p}\big(1-\mathrm{e}^{-pt/m}\big) - \frac{mg}{p}t\,.$$

The height $y{=}y(t)$ satisfies the initial-value problem

$$y'=v, \qquad y(0)=0.$$

The above function satisfies the second condition because

$$y(0) = \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{p}\big(1-\mathrm{e}^{-p\cdot 0/m}\big) - \frac{mg}{p}0 = \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{p}(1-1) = 0.$$

It satisfies the differential equation because

$$y'(t) = \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{p}\big(-\mathrm{e}^{-pt/m}\big)\cdot\frac{-p}{m} - \frac{mg}{p} = \bigg(v_0+\frac{mg}{p}\bigg)\mathrm{e}^{-pt/m} - \frac{mg}{p} = v(t);$$

so LHS of the differential equation equals to RHS of the differential equation when we plug in the above function $y{=}y(t)$.

Alternatively, we can first find the general solution of the differential equation and then the particular solution satisfying the initial condition:

$$\begin{aligned}
&y' = v = \bigg(v_0+\frac{mg}{p}\bigg)\mathrm{e}^{-pt/m} - \frac{mg}{p} \\
&\quad\Longrightarrow\quad y(t) = \int\bigg(\bigg(v_0+\frac{mg}{p}\bigg)\mathrm{e}^{-pt/m} - \frac{mg}{p}\bigg)\mathrm{d}t = \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{-p}\mathrm{e}^{-pt/m} - \frac{mg}{p}t + C,
\end{aligned}$$

where C is any constant. We now need to find C so that the function y defined above satisfies the initial condition $y(0){=}0$. For this, plug in $(t,y)=(0,0)$:

$$\begin{aligned}
0 &= \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{-p}\mathrm{e}^{-p\cdot 0/m} - \frac{mg}{p}0 + C = -\frac{m}{p}\bigg(v_0+\frac{mg}{p}\bigg) + C \quad\Longrightarrow\quad C=\frac{m}{p}\bigg(v_0+\frac{mg}{p}\bigg) \\
y(t) &= \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{-p}\mathrm{e}^{-pt/m} - \frac{mg}{p}t + \frac{m}{p}\bigg(v_0+\frac{mg}{p}\bigg) = \bigg(v_0+\frac{mg}{p}\bigg)\frac{m}{p}\big(1-\mathrm{e}^{-pt/m}\big) - \frac{mg}{p}t,
\end{aligned}$$

as claimed in the statement of the problem.

An even quicker way to obtain $y(t)$ is to use the Fundamental Theorem of Calculus. Since $y'(t)=v$ and $y(0)=0$,

$$\begin{aligned} y(t) = y(0) + \int_0^t y'(s)\mathrm{d}s &= \int_0^t \left(\left(v_0 + \frac{mg}{p}\right)\mathrm{e}^{-ps/m} - \frac{mg}{p}\right)\mathrm{d}s \\ &= \left(\left(v_0 + \frac{mg}{p}\right)\frac{m}{-p}\mathrm{e}^{-ps/m} - \frac{mg}{p}s\right)\Bigg|_{s=0}^{s=t} = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - \mathrm{e}^{-pt/m}\right) - \frac{mg}{p}t. \end{aligned}$$

(c; **5pts**) *Show that the amount of time the ball takes to reach the maximum height is*

$$t_1 = \frac{m}{p}\ln\left(\frac{mg + pv_0}{mg}\right).$$

Find this time if the mass of the ball is 1 kg, the initial speed is 20 m/s, and the air resistance is .1 kg/s.

The ball reaches its maximum height at the first time $t_1 > 0$ so that

$$0 = y'(t_1) = v(t_1) = \left(v_0 + \frac{mg}{p}\right)\mathrm{e}^{-pt_1/m} - \frac{mg}{p}.$$

This gives

$$\begin{aligned} \left(v_0 + \frac{mg}{p}\right)\mathrm{e}^{-pt_1/m} = \frac{mg}{p} \quad \Longrightarrow \quad \frac{pv_0 + mg}{mg} = \mathrm{e}^{pt_1/m} \quad &\Longrightarrow \quad \ln\left(\frac{mg + pv_0}{mg}\right) = \frac{pt_1}{m} \\ &\Longrightarrow \quad t_1 = \frac{m}{p}\ln\left(\frac{mg + pv_0}{mg}\right). \end{aligned}$$

Taking $m=1$ kg, $v_0=20$ m/s, $p=1/10$ kg/s, and $g=10$ m/s^2, we obtain

$$t_1 = \frac{1}{1/10}\ln\left(\frac{1\cdot 10 + (1/10)\cdot 20}{1\cdot 10}\right) = 10\ln 1.2 \approx 1.83 \text{ sec}$$

(d; **8pts**) *Show that*

$$y(2t_1) = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2\ln x\right)$$

where $x = \mathrm{e}^{pt_1/m}$ *and that for* $x > 1$ *the function*

$$f(x) = x - \frac{1}{x} - 2\ln x$$

is increasing. Use this result to decide whether $y(2t_1)$ *is positive or negative. What can you conclude from this? Does the ascent or descent take longer?*

By the statements of parts (2) and (3),

$$y(2t_1)=\left(v_0+\frac{mg}{p}\right)\frac{m}{p}\left(1-\mathrm{e}^{-p\cdot 2t_1/m}\right)-\frac{mg}{p}2t_1=\frac{m^2g}{p^2}\left(\frac{pv_0+mg}{mg}\left(1-(\mathrm{e}^{pt_1/m})^{-2}\right)-2\frac{pt_1}{m}\right)$$
$$=\frac{m^2g}{p^2}\left(x\left(1-x^{-2}\right)-2\ln x\right)=\frac{m^2g}{p^2}\left(x-\frac{1}{x}-2\ln x\right),$$

where $x=\mathrm{e}^{pt_1/m}$.

In order to see that f is increasing, take its derivative:

$$f'(x)=1+\frac{1}{x^2}-2\frac{1}{x}=\left(1-1/x\right)^2>0 \quad \text{if} \quad x>1.$$

Thus, for all $x>1$

$$f(x)>f(1)=1-\frac{1}{1}-2\ln 1=1-1-2\cdot 0=0.$$

In particular, if

$$x=\mathrm{e}^{pt_1/m}=\mathrm{e}^{\ln\left(\frac{mg+pv_0}{mg}\right)}=\frac{mg+pv_0}{mg}>1,$$

then

$$y(2t_1)=\frac{m^2g}{p^2}f\left(\mathrm{e}^{pt_1/m}\right)>0,$$

i.e. the ball is still above the ground at time $2t_1$. Thus, it takes longer to go down than to go up.

(e; **4pts**) *What is the answer to the last question in (d) if $p=0$ (no air resistance) and why?*
If there is no air resistance, $v=v_0-gt$ and $y=v_0t-\frac{1}{2}gt^2$ until the ball returns to the ground. It reaches the maximum height at the time t_1 when $v=0$, i.e. $t_1=v_0/g$, and hits the ground at the time $t_2>0$ when $y=0$, i.e. $t_2=2v_0/g$. Since $t_2=2t_1$, it takes as long to go up as to go down.

(f; **6pts**) *Is your answer in (e) consistent with the formula for $y(2t_1)$ in (d) and why?*
Take the limit of the expression for $y(2t_1)$ in (5) above as $p\longrightarrow 0$:

$$\lim_{p\longrightarrow 0}y(2t_1)=\lim_{p\longrightarrow 0}\frac{m^2g}{p^2}\left(\mathrm{e}^{\frac{pt_1}{m}}-\frac{1}{\mathrm{e}^{\frac{pt_1}{m}}}-2\ln\mathrm{e}^{\frac{pt_1}{m}}\right)=m^2g\lim_{p\longrightarrow 0}\frac{\frac{mg+pv_0}{mg}-\frac{mg}{mg+pv_0}-2\ln\left(\frac{mg+pv_0}{mg}\right)}{p^2}.$$

Since the numerator and denominator of the last fraction above approach 0 as $p\longrightarrow 0$, we can apply l'Hospital rule, differentiating with respect to p:

$$\lim_{p\longrightarrow 0}\frac{y(2t_1)}{m^2g}=\lim_{p\longrightarrow 0}\frac{\frac{mg+pv_0}{mg}-\frac{mg}{mg+pv_0}-2\ln\left(\frac{mg+pv_0}{mg}\right)}{p^2}=\lim_{p\longrightarrow 0}\frac{\frac{v_0}{mg}+\frac{mgv_0}{(mg+pv_0)^2}-2\frac{v_0}{mg+pv_0}}{2p}$$
$$=\frac{v_0}{2mg}\lim_{p\longrightarrow 0}\frac{1+\frac{m^2g^2}{(mg+pv_0)^2}-2\frac{mg}{mg+pv_0}}{p}=\frac{v_0}{2mg}\lim_{p\longrightarrow 0}\frac{\left(1-\frac{mg}{mg+pv_0}\right)^2}{p}$$
$$=\frac{v_0}{2mg}\lim_{p\longrightarrow 0}\frac{\frac{p^2v_0^2}{(mg+pv_0)^2}}{p}=\frac{v_0}{2mg}\lim_{p\longrightarrow 0}\frac{pv_0^2}{(mg+pv_0)^2}=0$$

Thus, as $p\longrightarrow 0$ (the air resistance falls to 0), $y(2t_1)\longrightarrow 0$, and so the time t_2 at which the ball returns to the ground approaches twice the time t_1 at which it reaches the maximum height (and therefore the ascent and descent times approach each other). So, the answer in (6) is consistent with (5).