MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 2 (90pts)

WebAssign Problem 1 (12pts)

Match the differential equation with its direction field (labeled (a)-(e) on pp377/378 in the book). Give reasons for your answer.

$$(84)\ y' = -3y, \quad (85)\ y' = -3x, \quad (86)\ y' = \mathrm{e}^x, \quad (87)\ y' = \frac{1}{2}y+x, \quad (88)\ y' = -xy.$$

The slopes y' in (84) are independent of x, and so do not change under horizontal shifts of the picture. This is the case only in (c).

The slopes y' in (85) are independent of y, and so do not change under vertical shifts of the picture. They are sometimes positive and sometimes negative. This is the case only in (e).

The slopes y' in (86) are independent of y, and so do not change under vertical shifts of the picture. They are always positive. This is the case only in (b).

The slopes y' in (88) are 0 (horizontal) for $x=0$ (the y-axis) and $y=0$ (the x-axis). This is the case only in (d).

This leaves only (a) for (87). As a check, note that the slopes y' vanish at $(0,0)$, $(-1,2)$, and $(1,-2)$. This is the case only in (a).

(84) c; (85) e; (86) b; (87) a; (88) d

WebAssign Problem 2 (23pts)

Use Euler's method with step size .1 to estimate $y(.5)$, where $y(x)$ is the solution of the initial-value problem

$$y' = y + xy, \quad y = y(x), \qquad y(0) = 1.$$

Since the initial value of x is 0 and the final value is .5, while the step size is $h=.1$, there are

$$n = \frac{x_{\text{final}} - x_{\text{initial}}}{h} = \frac{.5-0}{.1} = 5$$

steps to make. Thus, we need to find estimates y_1, y_2, y_3, y_4, y_5 for $y(x_i)$ with

$$x_i = x_{\text{initial}} + hi = 0 + \frac{1}{10}i = \frac{i}{10}\,.$$

They are obtained inductively by computing the slope $s_i = y'(x_i, y_i)$ at (x_i, y_i) and moving along the slope line from x_i to $x_{i+1} = x_i + h$ so that $y_{i+1} = y_i + s_i h$. The results of this computation are:

i	x_i	y_i	$s_i = y_i(1+x_i)$	$y_{i+1} = y_i + \frac{1}{10}s_i$
0	0	1	$1(1+0)=1$	$1+\frac{1}{10} = \frac{11}{10}$
1	$\frac{1}{10}$	$\frac{11}{10}$	$\frac{11}{10}\left(1+\frac{1}{10}\right) = \frac{121}{100}$	$\frac{11}{10} + \frac{121}{1000} = \frac{1221}{1000}$
2	$\frac{1}{5}$	$\frac{1221}{1000}$	$\frac{1221}{1000}\left(1+\frac{1}{5}\right) = \frac{3663}{2500}$	$\frac{1221}{1000} + \frac{3663}{25{,}000} = \frac{8547}{6250}$
3	$\frac{3}{10}$	$\frac{8547}{6250}$	$\frac{8547}{6250}\left(1+\frac{3}{10}\right) = \frac{111{,}111}{62{,}500}$	$\frac{8547}{6250} + \frac{111{,}111}{625{,}000} = \frac{965{,}811}{625{,}000}$
4	$\frac{2}{5}$	$\frac{965{,}811}{625{,}000}$	$\frac{965{,}811}{625{,}000}\left(1+\frac{2}{5}\right) = \frac{6{,}760{,}677}{3{,}125{,}000}$	$\frac{965{,}811}{625{,}000} + \frac{6{,}760{,}677}{31{,}250{,}000} = \boxed{\frac{55{,}051{,}227}{31{,}250{,}000}}$

See the previous problem for comments.

Note 1: Our estimate y_5 is an *under*-estimate for $y(.5)$ because $y''>0$ for all $x\in(0,.5)$ and for all solutions $y=y(x)$ used to estimate $y(.5)$, and so the tangent lines lie below the graphs. In order to see this, note that $y'=y(1+x) > 0$ if $y>0$ and $x\ge 0$. Thus, a solution $y=y(x)$ of the differential equation is increasing for all $x\ge x_0\ge 0$ if $y(x_0)>0$. For such a solution y,

$$y'' = (y+xy)' = y' + y + xy' = (y+xy) + y + x(y+xy) = y(2+2x+x^2) > 0 \quad \text{for all} \quad x \ge x_0,$$

as claimed above.

Note 2: Since the above initial-value problem involves a separable differential equation, it can be solved (please do so!) to find that the actual solution is

$$y(x) = \mathrm{e}^{x+\frac{x^2}{2}}.$$

Once this function is given, one can check directly that it solves the initial-value problem (please do this also!). The above formula for the solution gives

$$y(.5) = \mathrm{e}^{\frac{1}{2}+\frac{1}{8}} = \mathrm{e}^{\frac{5}{8}} \approx 1.868.$$

So the above estimate

$$\frac{55{,}051{,}227}{31{,}250{,}000} = 1.761639264$$

is not great despite of the relatively small step size; this is because the second derivative is large.

Problem 2.1 (10pts)

Sketch the direction field for the differential equation $y' = xy - x^2$, $y = y(x)$, and then sketch a solution curve through $(0,1)$. For the latter purposes, assume that this solution curve crosses the line $y=x$.

Bonus (5pts, all or nothing). *Prove that this solution curve crosses the line $y=x$ without using an explicit formula for this solution curve.*

The direction field can be obtained by computing the slopes $y' = xy - x^2 = x(y-x)$ at a number of points (x_i, y_j) and depicting these in the xy-plane by short line segments through each point (x_i, y_j) of the corresponding slope. The main features in this case are the following.

- The slopes are 0 on the y-axis ($x = 0$) and on the line $y = x$; these slopes are represented by horizontal line segments. These two lines, the y-axis and the line $y=x$, break the xy-plane into 4 segments.
- The slopes are positive in the top region (where $x>0$ and $y>x$, being above the line $y=x$) and in the bottom region (where $x<0$ and $y<x$). The slopes are negative in the right region (where $x>0$ and $y<x$) and in the left region (where $x<0$ and $y>x$).
- the magnitude of the slopes increases "away" (roughly speaking) from the two lines of 0-slopes.

As $(0, 1)$ lies on the y-axis (line of 0-slopes), the slope is 0 and so the tangent line to the solution curve is horizontal. Just to the left of $(0, 1)$, the slopes are negative, and so the solution curve descends to $(0, 1)$ from the left. As we trace it backwards (to the left), it ascends and keeps on ascending, since it stays in the left region (where the slopes are negative and we trace them backwards). In fact, as we move to the left x becomes more negative, while $y-x$ becomes more positive (because y increases as we move to the left), and so $y' = x(y-x)$ becomes more negative. So the solution curve ascends steeper and steeper as we move to the left (following it backwards).

Just to the right of $(0, 1)$, the slopes are positive, and so the solution curve ascends from $(0, 1)$ to the right. It continues to do so until it crosses the line $y=x$ (line of 0-slopes); at this crossing the curve must be horizontal. To the right of the crossing, the curve enters the right region, where the slopes are negative. It thus begins to descend. The rate of descent increases (the curve becomes steeper), as x increases, because $y' = x(y-x)$ becomes more negative (x becomes more positive, while y moves further down from the line $y=x$).

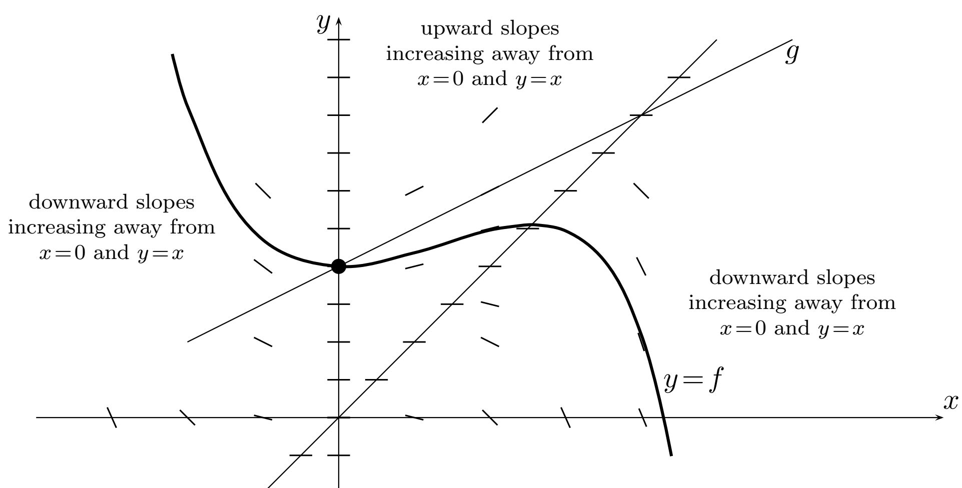


Figure 1: Graph of $y = y(x) = f(x)$ for Problem 2.1

The previous paragraph assumes that the curve actually crosses the line $y=x$. Is this really the case? Can't the curve just keep on going up in the top segment? In order to see that the curve should cross the line $y=x$, you need to draw the slopes at sufficiently small intervals to the right of $(0, 1)$. However convincing the resulting picture may be, this is not a proof (full justification), which is given below.

Bonus. Let $f=f(x)$ be the solution of $y'=x(y-x)$ with $f(0)=1$ and $g(x)=1+\frac{1}{2}x$. We want to show that the graph of f crosses the line $y=x$ for some $x>0$. Since the graph of g crosses this line at $(x,y)=(2,2)$, it is sufficient to show that $g(x)>f(x)$ for all $x>0$ (the graph of f then lies below the graph of g and thus must also cross the line $y=x$). Let $h(x)=g(x)-f(x)$. Then

$$h(0)=0, \qquad h'(x)=g'(x)-f'(x)=\frac{1}{2}-x\big(f(x)-x\big).$$

Thus, $h'(0)=\frac{1}{2}>0$ and so $h(x)>0$ for all $x>0$ small (since $h(0)=0$ and h is increasing near 0). If $x_1>0$ is the smallest number such that $h(x_1)=0$ (so f has climbed back to g), then $h'(c)=0$ for some $c\in(0,x_1)$ by the Mean Value Theorem of Calculus A. For this value of c, $h(c)=g(c)-f(c)>0$ and so

$$0=h'(c)=\frac{1}{2}-c\big(f(c)-c\big)>\frac{1}{2}-c\big(g(c)-c\big)=\frac{1}{2}-c\big(1+c/2-c\big)=\frac{1}{2}(c-1)^2\geq 0.$$

This is a contradiction, because it says $0>0$. So there is no $x_1>0$ such that $h(x_1)=0$, i.e. $g(x)=1+\frac{1}{2}x>f(x)$ for all $x>0$.

Note: Using Problem B, one can find that the solution to the above initial-value problem is given by

$$y(x)=\mathrm{e}^{x^2/2}-\mathrm{e}^{x^2/2}\int_0^x u^2\mathrm{e}^{-u^2/2}\mathrm{d}u.$$

Once given this solution, you can verify that it is indeed a solution (please do so!).

Problem 2.2 (10pts)

Make a rough sketch of the direction field for the autonomous differential equation $y'=f(y)$, where the graph of f is as shown in the first diagram in Figure 2. How does the limit behavior of solutions depend on the initial value of $y(0)$?

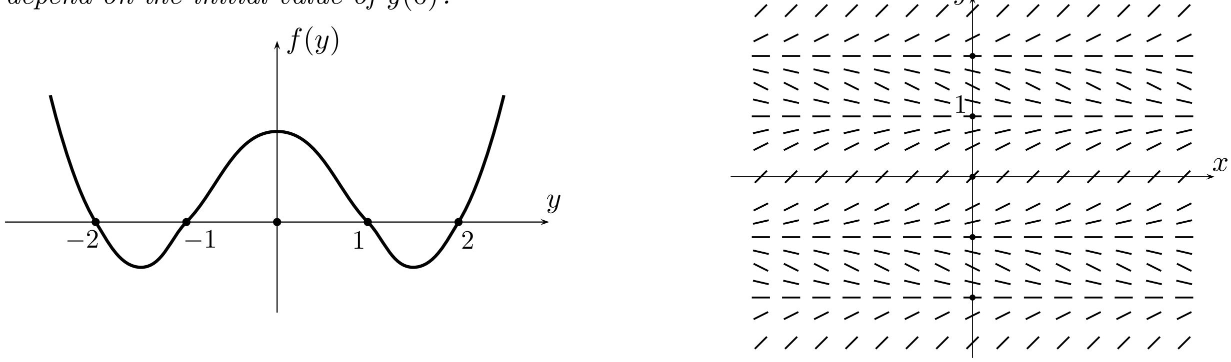


Figure 2: Graph of $f(y)$ for Problem 2.2 and direction field for $y'=f(y)$

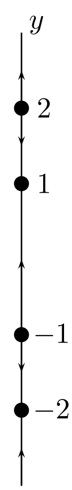
Since the slope $y'=f(y)$ does not depend on x, the direction field diagram does not change under horizontal shifts. Since the graph of f is symmetric about the f-axis, i.e. $f(-y)=f(y)$, the slopes do not change under the reflection about the x-axis, i.e. they are the same at (x,y) and $(x,-y)$. Since $f(y)=0$ for $y=\pm1,\pm2$, the slopes are horizontal on the four lines $y=\pm1,\pm2$, giving rise to four constant solutions.

Since $y'=f(y)>0$ for $y\in(-\infty,-2)\cup(-1,1)\cup(2,\infty)$, the direction fields in these regions are upward sloping and the solution curves ascend. In the region $y(0)\in(-\infty,-2)$, the slopes become steeper lower down and flatten near the line $y=-2$; thus, the graphs $y=y(x)$ descend rapidly

$x \longrightarrow -\infty$ and approach $y=-2$ as $x \longrightarrow \infty$. In the region $y(0) \in (-1,1)$, the slopes flatten near the lines $y=\pm 1$, while peaking on the line $y=0$; thus, the graphs approach $y=-1$ as $x \longrightarrow -\infty$ and $y=1$ as $x \longrightarrow \infty$. In the region $y(0) \in (2,\infty)$, the slopes become steeper higher up and flatten near the line $y=2$; thus, the graphs ascend rapidly as $x \longrightarrow \infty$ and approach $y=2$ as $x \longrightarrow -\infty$.

Since $y' = f(y) < 0$ for $y \in (-2,-1) \cup (1,2)$, the direction fields in these regions are downward sloping and the solution curves descend. In the region $y(0) \in (-2,-1)$, the slopes flatten near the lines $y=-2$ and $y=-1$; thus, the graphs approach the lines $y=-1$ as $x \longrightarrow -\infty$ and $y=-2$ as $x \longrightarrow \infty$. In the region $y(0) \in (1,2)$, the slopes flatten near the lines $y=1$ and $y=2$; thus, the graphs approach the lines $y=2$ as $x \longrightarrow -\infty$ and $y=1$ as $x \longrightarrow -\infty$.

The above conclusions are restated in Figure 3. The first diagram shows the y-axis and indicates the constant solutions by large dots. The arrows between the dots indicate whether the solutions in each interval go up or down. The second diagram shows a number of solution curves. The arrows in each region created by the constant solutions indicate whether the graphs of the other solutions in each segment rise or fall; these graphs must approach the two nearby constant solutions as $x \longrightarrow \pm\infty$. The first diagram makes sense only for *autonomous* first-order differential equations, i.e. equations of the form $y' = f(y)$, where y is a function of some variable x (note that there is no explicit dependence on x in the equation). The diagram of graphs is symmetric about the origin if the graph of $f(y)$ is symmetric about the y-axis (i.e. y is an even function, $f(-y) = f(y)$), as it seems to be. Figure 3 does not indicate direction fields; only solution curves are shown. The direction fields are tangent to these curves.



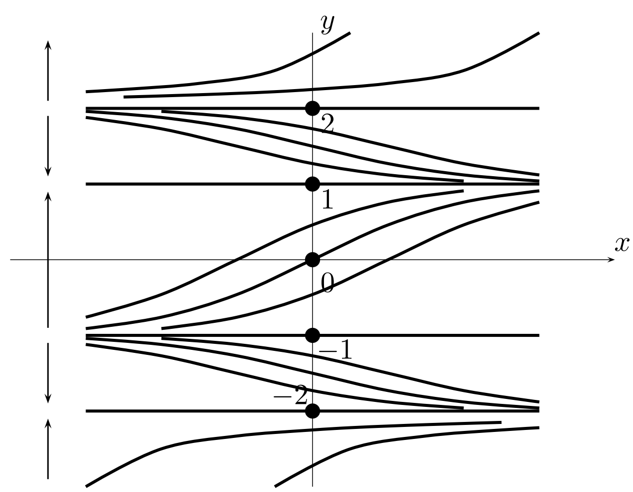


Figure 3: Diagrams of solution curves for Problem 2.2

Problem 2.3 (15pts)

Use Euler's method with step size .5 to compute approximate y-values y_1, y_2, y_3, and y_4 of the solution of the initial-value problem

$$y' = y - 2x, \quad y = y(x), \qquad y(1) = 0.$$

Use simple fractions p/q; no rounding.

Bonus (3pts, all or nothing). *Is your estimate y_4 for $y(x_4)$ an over- or under-estimate. Justify* *your answer without solving the IVP explicitly.*

These four values y_i are approximations for $y(x_i)$ with $x_i = x_0 + hi$, where $h = .5$. They are obtained inductively by computing the slope $s_i = y'(x_i, y_i)$ at (x_i, y_i) and moving along the slope line from x_i to $x_{i+1} = x_i + h$ so that $y_{i+1} = y_i + s_i h$. The results of this computation are:

i	x_i	y_i	$s_i = y_i - 2x_i$	$y_{i+1} = y_i + \frac{1}{2}s_i$
0	1	0	$0 - 2 = -2$	$0 - 1 = -1$
1	$\frac{3}{2}$	-1	$-1 - 3 = -4$	$-1 - 2 = -3$
2	2	-3	$-3 - 4 = -7$	$-3 - \frac{7}{2} = -\frac{13}{2}$
3	$\frac{5}{2}$	$-\frac{13}{2}$	$-\frac{13}{2} - 5 = -\frac{23}{2}$	$-\frac{13}{2} - \frac{23}{4} = \boxed{-\frac{49}{4}}$

The first column consists of the numbers i running from 0 to $n-1$, where n is the number of steps (4 in this case). The second column starts with the initial value of x (1 in this case) with subsequent entries in the column obtained by adding the step size h ($\frac{1}{2}$ in this case); it ends just before the final value of x would have been entered ($3 = \frac{5}{2} + \frac{1}{2}$ in this case). Thus, the first two columns can be filled in quickly at the start. The first entry in the third column is the initial y-value (0 in the case). After this, one computes the first entries in the remaining two columns and copies the first entry in the last column to the third column in the next line. The process then repeats across the second row and soon on. The estimate for the final value of y is the last entry in the table ($-\frac{49}{4}$ in this case).

$$\boxed{y_1 = -1,\ y_2 = -3,\ y_3 = -\frac{13}{2},\ y_4 = -\frac{49}{4}}$$

Note 1: While the table above provides a quick way of implementing Euler's method, it is important to understand what Euler's method does geometrically. Figure 4 shows the graph of the solution y to the initial-value problem; it *must* pass through $(1,0)$ due to the initial condition $y(0) = 1$. Figure 4 also shows the path corresponding to Euler's method with 4 steps going from $x_0 = 1$ to $x_f = 3$. Instead of following the solution curve (which we do not know), it follows the slope line from $(x_0, y_0) = (1, 0)$ until the x-value becomes $x_1 = 3/2$; we know the slope $s_0 = y_0 - 2x_0$ of this line and so can find the y-value y_1 corresponding to the x-value of $x_1 = 3/2$ on this slope line. Once we arrive at $(x_1, y_1) = (3/2, -1)$, we next move along the slope line through (x_1, y_1) until the x-value hits $x_2 = 2$; its slope is $s_1 = y_1 - 2x_1$. Once the y-value y_2 corresponding to the x-value of $x_2 = 2$ on this slope line is determined, this procedure is repeated twice more. Each of the four short directed line segments in the diagram corresponds to a row in the table.

Bonus. Our estimate y_4 is an *over*-estimate for $y(3)$ because $y'' < 0$ for all $x > 0$ and for all solutions $y = y(x)$ used to estimate $y(3)$, and so the tangent lines lie above the graphs. In order to see this, note that $y' = y - 2x < 0$ if $y < 0$ and $x \ge 0$. Thus, a solution $y = y(x)$ of the differential equation is decreasing for all $x \ge x_0 \ge 0$ if $y(x_0) < 0$. For such a solution y,

$$y'' = (y - 2x)' = y' - 2 = y - 2x - 2 < 0 - 2 \cdot x_0 - 2 < 0 \qquad \text{for all} \quad x \ge x_0,$$

as claimed above.

Note 2: The actual solution to this initial-value problem is $y(x) = 2 + 2x - 4e^{x-1}$, which can be obtained from Problem B and verified directly (please do so!). So $y(3) = 8 - 4e^2 \approx -21.56$, while

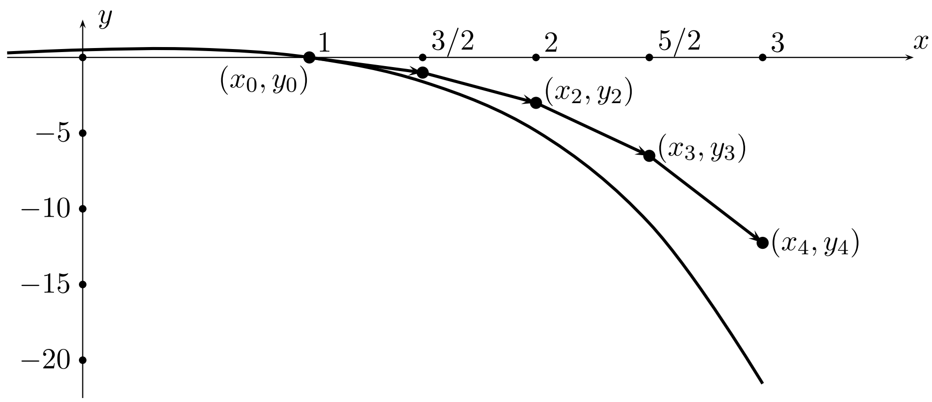


Figure 4: Graph of solution to IVP in Problem 2.3 and its approximation by 4-step Euler's method

$-49/4 = -12.25$. So the estimate is not so good, which happened because the solution decreases rapidly and the step size h was rather large.

Note 3: There will be an Euler's method problem on the first midterm (and possibly on the final). No calculators will be allowed. You'll need to carry out the computations using simple fractions (e.g. $-\frac{23}{2}$) and simplify your final answer as much as possible (e.g. $\frac{6}{48} = \frac{1}{8}$).

Problem B (20pts)

The Fundamental Theorem of Calculus provides a quick way of finding the general solution to differential equations of the form

$$y' = f(x), \qquad y = y(x). \tag{1}$$

It turns out that every equation of the form

$$y' + a(x)y = f(x), \qquad y = y(x), \tag{2}$$

can be reduced to (1). Simply multiply both sides of (2) by a nonzero function $h = h(x)$ *such that* $h' = ah$*:*

$$h(x)y' + a(x)h(x)y = h(x)f(x) \quad \Longleftrightarrow \quad hy' + h'y = hf \quad \Longleftrightarrow \quad (hy)' = hf.$$

We can integrate both sides of the last equation and then divide by h*.*

(a; **2pts**) *Show that for any function* $a = a(x)$*, there exists a nonzero function* $h = h(x)$ *such that* $h' = ah$*.*
Let

$$f(x) = \int_c^x a(u)\mathrm{d}u, \qquad h(x) = \mathrm{e}^{f(x)},$$

where c is any fixed constant (e.g. 0). By HW1 Problem A, part (i), $h' = ah$.

(b; **8pts**) *Find the general solution of the differential equation*

$$y' + 2y = 2\mathrm{e}^x, \qquad y = y(x).$$

What is the relation of this solution with the function $y(x)=\frac{2}{3}\mathrm{e}^x+\mathrm{e}^{-2x}$ *?*

In this case, $a(x)=2$ for all x and so we can take

$$h(x) = \mathrm{e}^{\int_0^x 2\mathrm{d}u} = \mathrm{e}^{2x}.$$

By the product rule, our differential equation is equivalent to

$$\big(\mathrm{e}^{2x}y\big)' = \mathrm{e}^{2x}\cdot 2\mathrm{e}^x \qquad \Longrightarrow \qquad \mathrm{e}^{2x}y(x) = \frac{2}{3}\mathrm{e}^{3x}+C \qquad \Longrightarrow \qquad \boxed{y(x) = \tfrac{2}{3}\mathrm{e}^x + C\mathrm{e}^{-2x}}$$

The function $y(x) = \frac{2}{3}\mathrm{e}^x+\mathrm{e}^{-2x}$ is a solution of $y'+2y=2\mathrm{e}^x$. It corresponds to the $C=1$ case of the general solution above.

(c; **10pts**) *Find the solution to the initial value problem*

$$y' = x+y, \qquad y=y(x), \qquad y(0)=1.$$

Move y to LHS and multiply both sides by e^{-x}:

$$\begin{aligned} e^{-x}y' - e^{-x}y = \mathrm{e}^{-x}x \qquad &\Longrightarrow \qquad \big(e^{-x}y\big)' = \mathrm{e}^{-x}x \\ &\Longrightarrow \qquad e^{-x}y(x) = \int e^{-x}x\mathrm{d}x = -\int x\mathrm{d}\mathrm{e}^{-x} = -\Big(x\mathrm{e}^{-x} - \int \mathrm{e}^{-x}\mathrm{d}x\Big) \\ &\qquad\qquad\qquad\quad = -x\mathrm{e}^{-x} - \mathrm{e}^{-x} + C \\ &\Longrightarrow \qquad y(x) = -1-x+C\mathrm{e}^x. \end{aligned}$$

This is the general solution of the differential equation. The initial condition gives

$$1 = -1-0+C\mathrm{e}^0 \qquad \Longrightarrow \qquad C=2.$$

So the solution to the above initial-value problem is $\boxed{y(x) = -1-x+2\mathrm{e}^x}$ In particular,

$$y(1) = -2+2\mathrm{e}^1 \approx 3.436564.$$

Remark: One of the aims of this problem is to introduce another collection of first-order differential equation that can be solved explicitly. One such a collection, consisting of *separable* equations $y'=f(x)g(y)$, is introduced in §4.3. The collection (2), introduced is this problem is called *linear* first-order differential equations; they are treated in detail in §4.5, which is not part of this course. While the two collections are very different, the differential equations (1) are both linear (take $a(x)=0$ in (2)) and separable (take $g(y)=1$). So are the linear equations (2) with $f(x)=0$ as well as with $a(x)$ and $f(x)$ both constant (the latter ones are also autonomous, with no explicit dependence on x). All of these equations can thus be solved in two different ways.