MAT 127: Calculus C, Spring 2022
Homework Assignment 2

WebAssign Problems due before 9am, Wednesday, 02/09
20% bonus for submissions **before 9am, Saturday, 02/05**

Written Assignment due before 4pm, Wendesday, 02/09
in your instructor's office

Please read Section 4.2 thoroughly before starting on the problem set; looking over Section 7.2 of the WebAssign textbook may be helpful too.

Written Assignment: 2.1,2.2,2.3,B (below and next page)
Show your work; correct answers without explanation will receive no credit, unless noted otherwise

Please write your solutions legibly; the graders will disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first), lecture number (L01, L02, or L03), and HW number in the upper-right corner of the first page.

Problem 2.1

Sketch the direction field for the differential equation $y' = xy - x^2$, $y = y(x)$, and then sketch a solution curve through $(0, 1)$. For the latter purposes, assume that this solution curve crosses the line $y=x$.
Bonus (all or nothing). *Prove that this solution curve crosses the line $y=x$ without using an explicit formula for this solution curve.*

Problem 2.2

Make a rough sketch of the direction field for the autonomous differential equation $y'=f(y)$, where the graph of f is as shown in Figure 1 below. How does the limit behavior of solutions depend on the initial value of $y(0)$?

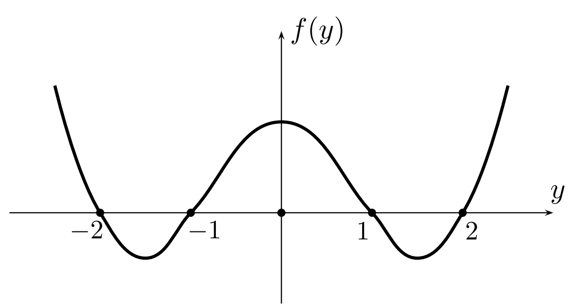

Figure 1: Graph of $f(y)$ for Problem 2.2

Problem 2.3

Use Euler's method with step size .5 to compute approximate y-values y_1, y_2, y_3, and y_4 of the solution of the initial-value problem

$$y' = y - 2x, \quad y = y(x), \qquad y(1) = 0.$$

Use simple fractions p/q; no rounding.
Bonus (all or nothing). Is your estimate y_4 for $y(x_4)$ an *over-* or *under*-estimate. Justify your answer without solving the IVP explicitly.

Problem B

The Fundamental Theorem of Calculus from *Calculus B* provides a quick way of finding the general solution to a differential equations of the form

$$y' = f(x), \qquad y = y(x). \tag{1}$$

It turns out that every equation of the form

$$y' + a(x)y = f(x), \qquad y = y(x), \tag{2}$$

can be reduced to (1). Simply multiply both sides of (2) by a nonzero function $h = h(x)$ such that $h' = ah$:

$$h(x)y' + a(x)h(x)y = h(x)f(x) \iff hy' + h'y = hf \iff (hy)' = hf.$$

We can integrate both sides of the last equation and then divide by h. For example, multiplying

$$y' + 2xy = 2x, \qquad y = y(x), \tag{3}$$

by $h(x) = e^{x^2}$ gives

$$\mathrm{e}^{x^2}y' + 2x\mathrm{e}^{x^2}y = 2x\mathrm{e}^{x^2} \iff \left(\mathrm{e}^{x^2}y\right)' = 2x\,\mathrm{e}^{x^2} \iff \mathrm{e}^{x^2}y = \int 2x\,\mathrm{e}^{x^2}\mathrm{d}x = \mathrm{e}^{x^2} + C.$$

So the general solution of the differential equation (3) is $y(x) = 1 + Ce^{-x^2}$.

(a) Show that for any function $a = a(x)$, there exists a nonzero function $h = h(x)$ such that $h' = ah$.
Hint: see HW1 Problem A.

(b) Find the general solution of the differential equation

$$y' + 2y = 2\mathrm{e}^x, \qquad y = y(x).$$

What is the relation of this solution with the function $y(x) = \frac{2}{3}\mathrm{e}^x + \mathrm{e}^{-2x}$?
Hint for 1st part: comparing this equation with equation (2) above, what are the functions $a(x)$, $f(x)$, and $h(x)$ here? How does the sentence following equation (2) apply in this case?

(c) Find the solution to the initial value problem

$$y' = x + y, \qquad y = y(x), \qquad y(0) = 1.$$