MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 12 (105pts)

WebAssign Problem 1 (8pts)

Use multiplication and division to find the first three nonzero terms in the Taylor series

$$y = \mathrm{e}^{-x^2}\cos x \qquad \text{at} \quad x=0$$

Since

$$\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}\,, \quad \mathrm{e}^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}\,, \qquad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\,,$$

we obtain

$$\begin{aligned}
\mathrm{e}^{-x^2}\cos x &= \bigg(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}\bigg)\bigg(\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}\bigg) = \Big(1-\frac{1}{1!}x^2+\frac{1}{2!}x^4-\ldots\Big)\Big(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\ldots\Big)\\
&= 1+\bigg(1\cdot\Big(-\frac{1}{2!}x^2\Big)+\Big(-\frac{1}{1!}x^2\Big)\cdot 1\bigg)+\bigg(1\cdot\frac{1}{4!}x^4+\Big(-\frac{1}{1!}x^2\Big)\Big(-\frac{1}{2!}x^2\Big)+\frac{1}{2!}x^4\cdot 1\bigg)+\ldots\\
&= 1-\Big(\frac{1}{2}+1\Big)x^2+\Big(\frac{1}{24}+\frac{1}{2}+\frac{1}{2}\Big)x^4+\ldots = \boxed{1-\frac{3}{2}x^2+\frac{25}{24}x^4}+\ldots
\end{aligned}$$

Note: if one of the above coefficients turned out to be 0, we would have to expand each factor another step to get three *nonzero* terms of the product.

WebAssign Problem 2 (18pts)

(a) *Expand* $1/\sqrt[4]{1+x}$ *as a power series.*
(b) *Use it to estimate* $1/\sqrt[4]{1.1}$ *correct to 3 decimal place.*

(a) By the binomial theorem:

$$1/\sqrt[4]{1+x} = (1+x)^{-1/4} = \sum_{n=0}^{\infty}\binom{-1/4}{n}x^n\,, \qquad \text{where}$$

$$\binom{-1/4}{n} = \frac{(-1/4)(-1/4-1)\ldots\cdot(-1/4-n+1)}{n!} = \frac{(-1)^n\frac{1}{4}\cdot\frac{5}{4}\cdot\ldots\cdot\frac{4n-3}{4}}{n!} = (-1)^n\frac{1\cdot 5\cdot\ldots\cdot(4n-3)}{4^n n!}.$$

So $1/\sqrt[4]{1+x} = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 5\cdot\ldots\cdot(4n-3)}{4^n n!}x^n}$ By the binomial series theorem, this holds if $|x|<1$.

(b) Since $|.1|<1$, by part (a)

$$1/\sqrt[4]{1.1} = 1/\sqrt[4]{1+.1} = \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 5\cdot\ldots\cdot(4n-3)}{4^n n!}.1^n = \sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 5\cdot\ldots\cdot(4n-3)}{4^n n!\cdot 10^n}.$$

This infinite sum is to be estimated by the sum of the first m terms with m chosen so that

$$\bigg|\sum_{n=0}^{\infty}(-1)^n\frac{1\cdot 5\cdot\ldots\cdot(4n-3)}{4^n n!\cdot 10^n} - \sum_{n=0}^{n=m}(-1)^n\frac{1\cdot 5\cdot\ldots\cdot(4n-3)}{4^n n!\cdot 10^n}\bigg| < \frac{1}{2}\cdot 10^{-3}.$$

This series is alternating, with

$$b_{n+1} = \frac{1\cdot 5\cdot \ldots \cdot (4n-3)\cdot (4(n+1)-3)}{4^{n+1}(n+1)!\cdot 10^{n+1}} = \frac{4n+1}{4(n+1)10} b_n < \frac{b_n}{10}.$$

Thus, $b_{n+1} < b_n$ and $b_n \longrightarrow 0$. So, by the Alternating Series Estimation Theorem on p587,

$$\left| \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 5\cdot \ldots \cdot (4n-3)}{4^n n!\cdot 10^n} - \sum_{n=0}^{n=m} (-1)^n \frac{1\cdot 5\cdot \ldots \cdot (4n-3)}{4^n n!\cdot 10^n} \right| < b_{m+1} = \frac{1\cdot 5\cdot \ldots \cdot (4m-3)\cdot (4(m+1)-3)}{4^{m+1}(m+1)!\cdot 10^{m+1}}$$

So it is enough to choose m so that

$$\frac{1\cdot 5\cdot \ldots \cdot (4m-3)\cdot (4m+1)}{4^{m+1}(m+1)!\cdot 10^{m+1}} < \frac{1}{2\cdot 10^3}$$

Considering the exponents of 10 above, the first m to try is $m=2$:

$$\frac{1\cdot 5\cdot 9}{4^{2+1}(2+1)!\cdot 10^{2+1}} = \frac{45}{64\cdot 6\cdot 10^3} < \frac{1}{2\cdot 10^3}$$

So $m=2$ works. We could also try $m=1$, but

$$\frac{1\cdot 5}{4^{1+1}(1+1)!\cdot 10^{1+1}} = \frac{5}{16\cdot 2\cdot 10^2} > \frac{1}{2\cdot 10^3}.$$

So, we must use the finite sum estimate with $m=2$:

$$1/\sqrt[4]{1.1} \approx \sum_{n=0}^{n=2} (-1)^n \frac{1\cdot 5\cdot \ldots \cdot (4n-3)}{4^n n!\cdot 10^n} = 1 - \frac{1}{4^1 1!\cdot 10^1} + \frac{1\cdot 5}{4^2 2!\cdot 10^2} = 1 - \frac{1}{4\cdot 10} + \frac{5}{32\cdot 10^2}$$

$$= 1 - \frac{1}{4\cdot 10} + \frac{1}{32\cdot 2\cdot 10} = \frac{640-16+1}{640} = \frac{625}{640} = \boxed{\frac{125}{128}}$$

This is an *over*-estimate because the last term used is positive (this reasoning is only for *alternating* series; for sums with only positive terms, a finite-sum estimate is always an *under*-estimate).

Remark: $1/\sqrt[4]{1.1} \approx .9765$, while $125/128 \approx .9766$, so our estimate is indeed accurate to 3 decimal places and is an over-estimate.

WebAssign Problem 24 (9pts)

Use the Alternating Series Test to estimate the range of values of x for which the approximation

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

is accurate to within .005.

For each fixed x, the infinite series $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ is alternating,

$$\frac{x^{2(n+1)}}{(2(n+1))!} = \frac{x^{2n}}{(2n)!}\cdot \frac{x^2}{(2n+1)(2n+2)} < \frac{x^{2n}}{(2n)!} \qquad \text{if} \quad 2n+1 \geq |x|,$$

and $x^{2n}/(2n)! \longrightarrow 0$ because the ratio of two consecutive terms approaches 0. Thus, by the Alternating Series Estimation Theorem on p587,

$$\left|\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\right| = \left|\cos x - \sum_{n=0}^{n=2} (-1)^n \frac{x^{2n}}{2n!}\right| < b_3 = \frac{x^{2\cdot 3}}{(2\cdot 3)!} = \frac{x^6}{720} \qquad \text{if} \quad |x| \le 7.$$

So, the estimate is accurate to within .005 if $x^6/720 \le 1/200$ or $\boxed{|x| \le \sqrt[6]{3.6}}$

Problem XII.1 (5pts)

Use the binomial series to expand the function $f(x) = \dfrac{1}{(1+x)^4}$ *as a power series centered at* $x=0$ *and state the radius of convergence.*

By the binomial theorem:

$$\frac{1}{(1+x)^4} = (1+x)^{-4} = \sum_{n=0}^{\infty} \binom{-4}{n} x^n, \qquad \text{where}$$

$$\binom{-4}{n} = \frac{(-4)(-4-1)\ldots(-4-n+1)}{n!} = \frac{(-1)^n 4\cdot 5\cdot \ldots (n+3)}{n!} = (-1)^n \frac{(n+1)(n+2)(n+3)}{6}.$$

So $$\frac{1}{(1+x)^4} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)(n+3)}{6} x^n = \sum_{n=0}^{\infty} (-1)^n \frac{n^3 + 6n^2 + 11n + 6}{6} x^n}$$

The radius of convergence is $\boxed{1}$ by the binomial series theorem.

Problem XII.2 (10pts)

(a) *Show that the function* $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ *satisfies*

$$g'(x) = \frac{kg(x)}{1+x} \qquad -1 < x < 1.$$

(b) *Show that the function* $h(x) = (1+x)^{-k} g(x)$ *satisfies* $h'(x) = 0$.
(c) *Conclude that* $g(x) = (1+x)^k$.

(a; **6pts**) Since the g series converges on $(-1,1)$, by the theorem on p600 on $(-1,1)$

$$g'(x) = \sum_{n=1}^{\infty} \binom{k}{n} n x^{n-1} = \sum_{n=1}^{\infty} \frac{k\cdot(k-1)\cdot\ldots\cdot(k-n+1)}{n!} n x^{n-1}$$
$$= k \sum_{n=1}^{\infty} \frac{(k-1)\cdot\ldots\cdot(k-n+1)}{(n-1)!} x^{n-1} = k \sum_{n=0}^{\infty} \frac{(k-1)\cdot\ldots\cdot(k-n)}{n!} x^n$$

This gives

$$\begin{aligned}\frac{(1+x)}{k}g'(x) &= \sum_{n=0}^{\infty}\frac{(k-1)\cdot\ldots\cdot(k-n)}{n!}x^n + x\sum_{n=1}^{\infty}\frac{(k-1)\cdot\ldots\cdot(k-n+1)}{(n-1)!}x^{n-1}\\ &= 1+\sum_{n=1}^{\infty}\frac{(k-1)\cdot\ldots\cdot(k-n+1)}{(n-1)!}\left(\frac{k-n}{n}+1\right)x^n\\ &= 1+\sum_{n=1}^{\infty}\frac{(k-1)\cdot\ldots\cdot(k-n+1)k}{(n-1)!n} = \sum_{n=0}^{\infty}\binom{k}{n}x^n = g(x).\end{aligned}$$

This proves the required identity.

(b; **3pts**) By the product rule and part (a), on $(-1,1)$

$$h'(x) = \big((1+x)^{-k}\big)'g(x) + (1+x)^{-k}g'(x) = -k(1+x)^{-k-1}g(x) + (1+x)^{-k}\frac{kg(x)}{1+x} = 0.$$

(c; **1pt**) By part (b), $h(x) = C$ for some constant C and so $g(x) = C(1+x)^k$ on $(-1,1)$. Since $g(0) = \binom{k}{0} = 1$ and $(1+0)^k = 1$, $C = 0$.

Problem XII.3 (20pts)

(a) *Suppose the power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges to a function* $y = y(x)$ *such that*

$$y''-y'+y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Find a formula that expresses a_{n+2} *in terms of* a_{n+1} *and* a_n *and determine* a_0, a_1, a_2, a_3.

(b) *Solve the initial-value problem in (a) exactly (find a simple formula for* $y = y(x)$*).*

(c) *Use your answer in (b), Taylor series, and multiplication of power series to recover the values of* a_0, a_1, a_2, a_3 *you found in (a).*

(a; **7pts**) If $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n,$$

$$y''-y'+y = \sum_{n=0}^{\infty}\big(a_n-(n+1)a_{n+1}+(n+2)(n+1)a_{n+2}\big)x^n.$$

From $y''-y'+y = 0$, we obtain $a_n-(n+1)a_{n+1}+(n+2)(n+1)a_{n+2} = 0$, i.e.

$$\boxed{a_{n+2} = \frac{1}{n+2}\left(a_{n+1} - \frac{1}{n+1}a_n\right)}$$

From $y(0)=1$ and $y'(0)=0$, we obtain $\boxed{a_0=1,\ a_1=0}$ Combining this with the above equation, we obtain $\boxed{a_2=-1/2,\ a_3=-1/6}$

(b; **7pts**) The differential equation in (a) is a second-order linear homogeneous ODE with constant coefficients. Its characteristic polynomial is r^2-r+1. The roots of this polynomial are $r=(1\pm \mathrm{i}\sqrt{3})/2$. Thus, the general solution of the ODE in (a) is

$$y(x) = C_1\mathrm{e}^{x/2}\cos(\sqrt{3}x/2)+C_2\mathrm{e}^{x/2}\sin(\sqrt{3}x/2).$$

We need to find C_1, C_2 so that

$$\begin{cases} y(0) = C_1\mathrm{e}^0\cos 0+C_2\mathrm{e}^0\sin 0 = 1 \\ y'(0) = \frac{1}{2}C_1\big(\mathrm{e}^0\cos 0-\sqrt{3}\sin 0\big)+\frac{1}{2}C_2\big(\mathrm{e}^0\sin 0+\sqrt{3}\cos 0\big) = 0 \end{cases} \qquad\Longleftrightarrow\qquad \begin{cases} C_1 = 1 \\ C_1+\sqrt{3}C_2 = 0 \end{cases}$$

Thus, $C_1=1$, $C_2=-\sqrt{3}/3$, and the solution to the initial-value problem in (a) is

$$\boxed{y(x) = \mathrm{e}^{x/2}\cos(\sqrt{3}x/2)-(\sqrt{3}/3)\mathrm{e}^{x/2}\sin(\sqrt{3}x/2)}$$

(c; **6pts**) We need to determine the first 4 terms (degree 0,1,2,3) in the Taylor series expansion of the solution $y=y(x)$ we found in (b). Using the power series

$$\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots,$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1-\frac{x^2}{2}+\ldots, \qquad \sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{6}+\ldots,$$

we obtain

$$\begin{aligned} y(x) &= \mathrm{e}^{x/2}\big(\cos(\sqrt{3}x/2)-(\sqrt{3}/3)\sin(\sqrt{3}x/2)\big) \\ &= \left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\ldots\right)\!\left(\!\left(1-\frac{3x^2}{8}+\ldots\right)-\frac{\sqrt{3}}{3}\left(\frac{\sqrt{3}x}{2}-\frac{3\sqrt{3}x^3}{48}+\ldots\right)\!\right) \\ &= 1+\left(\frac{1}{2}-\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2}\right)x+\left(\frac{1}{8}-\frac{1}{2}\cdot\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2}-\frac{3}{8}\right)x^2+\left(\frac{1}{48}-\frac{1}{8}\cdot\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{3}{8}+\frac{\sqrt{3}}{3}\cdot\frac{3\sqrt{3}}{48}\right)x^3+\ldots \\ &= 1+0x-\frac{1}{2}x^2-\frac{1}{6}x^3+\ldots \end{aligned}$$

Thus, the first 4 Taylor coefficients of the solution $y=y(x)$ to the initial-value problem in (a) are

$$\boxed{a_0=1,\ a_1=0,\ a_2=-1/2,\ a_3=-1/6}$$

as obtained in (a).

Problem L (35pts)

Let f_n be the n-th Fibonacci number as on page 445,

$$A_n = \sum_{k=1}^{k=n} k = 1+2+\ldots+n, \qquad B_n = \sum_{k=1}^{k=n} k^2 = 1^2+2^2+\ldots+n^2;$$

by definition $f_0 = A_0 = B_0 = 0$.

(a; **3pts**) *Give a recursive definition of the numbers f_n, A_n, B_n with $n \geq 0$.*

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+2} = f_{n+1} + f_n \quad n \geq 0,$$
$$A_0 = 0, \quad A_{n+1} = A_n + n+1 \quad n \geq 0, \qquad B_0 = 0, \quad B_{n+1} = B_n + (n+1)^2 \quad n \geq 0.$$

(b; **9pts**) Use mathematical induction and only part (a) to show that $f_n, A_n, B_n \leq 5^n$ for all $n \geq 0$.

This is true for $n=0,1$: ($f_0 = A_0 = B_0 = 0$, $f_1 = A_1 = B_1 = 1$). Suppose this is true for some $n \geq 1$. Then,

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} \leq 5^n + 5^{n-1} \leq 5^n + 5^n = 5^n \cdot 2 \leq 5^{n+1} \\ A_{n+1} &= A_n + n + 1 \leq A_n + A_n + A_n \leq 3 \cdot 5^n \leq 5^{n+1} \\ B_{n+1} &= B_n + (n+1)^2 \leq B_n + n^2 + 2n + 1 \leq B_n + B_n + 2B_n + B_n \leq 5 \cdot 5^n = 5^{n+1}. \end{aligned}$$

So, if $f_n, A_n, B_n \leq 5^n$, then $f_{n+1}, A_{n+1}, B_{n+1} \leq 5^{n+1}$. Thus, this is the case for all n.

(c; **3pts**) Use the *Absolute Convergence* and *Comparison* Tests and only part (b) to show that the power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad A(x) = \sum_{n=0}^{\infty} A_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} B_n x^n,$$

converge if $|x| < 1/6$ (and thus determine smooth functions near $x=0$).

Since $f_n, A_n, B_n \geq 0$, by the Absolute Convergence Test it is enough to show that the series

$$\sum_{n=0}^{\infty} \big|f_n x^n\big| = \sum_{n=0}^{\infty} f_n |x|^n, \quad \sum_{n=0}^{\infty} \big|A_n x^n\big| = \sum_{n=0}^{\infty} A_n |x|^n, \quad \sum_{n=0}^{\infty} \big|B_n x^n\big| = \sum_{n=0}^{\infty} B_n |x|^n$$

converge if $|x| < 1/6$. Since

$$0 \leq f_n|x|^n, A_n|x|^n, B_n|x|^n \leq \left(\frac{5}{6}\right)^n = \frac{5^n}{6^n} \qquad \text{if} \quad |x| \leq \frac{1}{6}$$

and the geometric series $\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$ converges, the claim follows from the Comparison Test.

(d; **10pts**) *Using only part (a), show that*

$$f(x) = x + xf(x) + x^2 f(x), \quad A(x) = xA(x) + \frac{x}{(1-x)^2}, \quad B(x) = xB(x) + \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

Since $f_0 = A_0 = B_0 = 0$,

$$\begin{aligned} f(x) = f_1x + \sum_{n=2}^{\infty} f_n x^n &= f_1 x + \sum_{n=2}^{\infty} \big(f_{n-1}+f_{n-2}\big)x^n = x + x\sum_{n=2}^{\infty} f_{n-1}x^{n-1} + x^2\sum_{n=2}^{\infty} f_{n-2}x^{n-2} \\ &= x + x\sum_{n=1}^{\infty} f_n x^n + x^2\sum_{n=0}^{\infty} f_n x^n = x + xf(x) + x^2 f(x) \end{aligned}$$

$$\begin{aligned} A(x) = \sum_{n=1}^{\infty} A_n x^n &= \sum_{n=1}^{\infty}(A_{n-1}+n)x^n = x\sum_{n=1}^{\infty} A_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} nx^{n-1} \\ &= x\sum_{n=0}^{\infty} A_n x^n + x\bigg(\sum_{n=0}^{\infty} x^n\bigg)' = xA(x) + x\bigg(\frac{1}{1-x}\bigg)' = xA(x) + x\frac{1}{(1-x)^2} \end{aligned}$$

$$\begin{aligned} B(x) = \sum_{n=1}^{\infty} B_n x^n &= \sum_{n=1}^{\infty}(B_{n-1}+n^2)x^n = x\sum_{n=1}^{\infty} B_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} nx^{n-1} + x^2\sum_{n=1}^{\infty} n(n-1)x^{n-2} \\ &= x\sum_{n=0}^{\infty} B_n x^n + x\bigg(\sum_{n=0}^{\infty} x^n\bigg)' + x^2\bigg(\sum_{n=0}^{\infty} x^n\bigg)'' = xB(x) + x\bigg(\frac{1}{1-x}\bigg)' + x^2\bigg(\frac{1}{1-x}\bigg)'' \\ &= xB(x) + x\frac{1}{(1-x)^2} + x^2\frac{2}{(1-x)^3} \end{aligned}$$

(e; **10pts**) *Using part (d), express f_n, A_n, and B_n explicitly in terms of n.*

By part (b),

$$\begin{aligned} A(x) = \sum_{n=0}^{\infty} A_n x^n &= \frac{x}{(1-x)^3} = \frac{x}{2}\cdot\bigg(\frac{1}{1-x}\bigg)'' = \frac{x}{2}\bigg(\sum_{n=0}^{\infty} x^n\bigg)'' = \frac{x}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} \\ &= \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-1} = \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n = \sum_{n=0}^{\infty}\frac{(n+1)n}{2}x^n \end{aligned}$$

$$\begin{aligned} B(x) = \sum_{n=0}^{\infty} B_n x^n &= \frac{x}{(1-x)^3} + \frac{2x^2}{(1-x)^4} = \frac{x}{2}\cdot\bigg(\frac{1}{1-x}\bigg)'' + \frac{2x^2}{6}\cdot\bigg(\frac{1}{1-x}\bigg)''' \\ &= \frac{x}{2}\bigg(\sum_{n=0}^{\infty} x^n\bigg)'' + \frac{2x^2}{6}\bigg(\sum_{n=0}^{\infty} x^n\bigg)''' = \frac{x}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} + \frac{2x^2}{6}\sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3} \\ &= \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-1} + \sum_{n=3}^{\infty}\frac{2n(n-1)(n-2)}{6}x^{n-1} = \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^n + \sum_{n=2}^{\infty}\frac{2(n+1)n(n-1)}{6}x^n \\ &= \sum_{n=0}^{\infty}\frac{(n+1)n}{2}x^n + \sum_{n=0}^{\infty}\frac{2(n+1)n(n-1)}{6}x^n = \sum_{n=0}^{\infty}\frac{(2n+1)(n+1)n}{6}x^n \end{aligned}$$

Comparing the left and right-most power series, we obtain

$$\boxed{A_n = 1+2+\ldots+n = \frac{(n+1)n}{2}, \qquad B_n = 1^2+2^2+\ldots+n^2 = \frac{(2n+1)(n+1)n}{6}}$$

Also, by part (b),

$$f(x) = \frac{x}{1-x-x^2} = -x\frac{1}{(x-r_-)(x+r_+)},$$

where $r_\pm = \dfrac{-1 \pm \sqrt{5}}{2}$ are the two roots of $x^2 + x - 1 = 0$; thus, $r_- r_+ = -1$ and $r_+ - r_- = \sqrt{5}$. Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$,

$$\frac{1}{x - r_-} = -\frac{1}{r_-(1 - x/r_-)} = \frac{r_+}{1 - (-r_+ x)} = r_+ \sum_{n=0}^{\infty} (-r_+ x)^n\,;$$
$$\frac{1}{x - r_+} = -\frac{1}{r_+(1 - x/r_+)} = \frac{r_-}{1 - (-r_- x)} = r_- \sum_{n=0}^{\infty} (-r_- x)^n\,.$$

We use these two Taylor series to get a Taylor series for $f(x) = x/(1 - x - x^2)$. This can be done using multiplication of power series:

$$\begin{aligned}
f(x) = \sum_{n=0}^{\infty} f_n x^n &= -x \cdot \frac{1}{x - r_-} \cdot \frac{1}{x - r_+} = -x \left(r_+ \sum_{n=0}^{\infty} (-r_+ x)^n \right) \left(r_- \sum_{n=0}^{\infty} (-r_- x)^n \right) \\
&= -x r_- r_+ \sum_{n=0}^{\infty} \big(1 \cdot (-r_- x)^n + (-r_+ x)(-r_- x)^{n-1} + \ldots + (-r_+ x)^{n-1}(-r_- x) + (-r_+ x)^n \cdot 1 \big) \\
&= x \sum_{n=0}^{\infty} \big(r_+^0 \cdot r_-^n + r_+ r_- + \ldots + r_+^{n-1} r_- + r_+^n \cdot r_-^0 \big) (-x)^n \\
&= x \sum_{n=0}^{\infty} \frac{r_+^{n+1} - r_-^{n+1}}{r_+ - r_-} (-x)^n = -\sum_{n=0}^{\infty} \frac{r_+^{n+1} - r_-^{n+1}}{\sqrt{5}} (-x)^{n+1} = -\sum_{n=1}^{\infty} \frac{r_+^n - r_-^n}{\sqrt{5}} (-x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-r_-)^n - (-r_+)^n}{\sqrt{5}} x^n\,.
\end{aligned}$$

An easier alternative is to use partial fractions and addition of power series:

$$\begin{aligned}
f(x) = \sum_{n=0}^{\infty} f_n x^n &= -x \cdot \frac{1}{x - r_-} \cdot \frac{1}{x - r_+} = -x \cdot \frac{1}{(-r_+) - (-r_-)} \left(\frac{1}{x - r_-} - \frac{1}{x - r_+} \right) \\
&= \frac{x}{\sqrt{5}} \left(r_+ \sum_{n=0}^{\infty} (-r_+ x)^n - r_- \sum_{n=0}^{\infty} (-r_- x)^n \right) = -\sum_{n=0}^{\infty} \frac{r_+^{n+1} - r_-^{n+1}}{\sqrt{5}} (-x)^{n+1} = -\sum_{n=1}^{\infty} \frac{r_+^n - r_-^n}{\sqrt{5}} (-x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-r_-)^n - (-r_+)^n}{\sqrt{5}} x^n\,.
\end{aligned}$$

In either case, we find that

$$f_n = \frac{(-r_-)^n - (r_+)^n}{\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right) = f_n}$$

This was the formula in Problem G on HW6.

Remark: In principle, there is a simpler way of obtaining the above formula for the sum of square using mathematical induction. This approach can be used to compute sums of higher powers as well, but recursively (to get a formula for the sum of 5th powers, you'll first need to get formulas for the sums of smaller power). The above approach via power series can be used to compute sums of higher powers also, but more directly.