MAT 127: Calculus C, Spring 2022
Homework Assignment 12

WebAssign Problems due before 9am, Wednesday, 05/04
20% bonus for submissions **before 9am, Saturday, 04/30**

Written Assignment due before 4pm, Wednesday, 05/04
in your instructor's office (L01 in Math 4-101B, L02/3 in Math 3-111)

Please read Section 6.4 thoroughly before starting on the problem set; looking over Sections 8.8 and 8.9 of the WebAssign textbook may be helpful too.

Written Assignment: Problems XII.1-3,L (below and next page)
Show your work; correct answers without explanation will receive no credit, unless noted otherwise.

Please write your solutions legibly; the graders will disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first), lecture number (L01, L02, or L03), and HW number in the upper-right corner of the first page.

Problem XII.1

Use the binomial series to expand the function $f(x) = \frac{1}{(1+x)^4}$ as a power series centered at $x=0$ and state the radius of convergence.

Problem XII.2

(a) Show that the function $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ satisfies

$$g'(x) = \frac{kg(x)}{1+x} \qquad -1 < x < 1.$$

(b) Show that the function $h(x) = (1+x)^{-k} g(x)$ satisfies $h'(x) = 0$.
(c) Conclude that $g(x) = (1+x)^k$.

Problem XII.3

(a) Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $y=y(x)$ such that

$$y'' - y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Find a formula that expresses a_{n+2} in terms of a_{n+1} and a_n and determine a_0, a_1, a_2, a_3.

(b) Solve the initial-value problem in (a) exactly (find a simple formula for $y=y(x)$).

(c) Use your answer in (b), Taylor series, and multiplication of power series to recover the values of a_0, a_1, a_2, a_3 you found in (a).

Problem L

Power series are typically used to "break" a function into a sequence of numbers (the Taylor coefficients of the function). However, sometimes it is useful to go in the opposite direction, assembling a sequence of numbers into a function.

Let f_n be the n-th Fibonacci number of Example 3c in 8.1,

$$A_n = \sum_{k=1}^{k=n} k = 1+2+\ldots+n, \qquad B_n = \sum_{k=1}^{k=n} k^2 = 1^2+2^2+\ldots+n^2;$$

by definition $f_0 = A_0 = B_0 = 0$.

(a) Give a recursive definition of the numbers f_n, A_n, B_n with $n \geq 0$.

(b) Use mathematical induction and only part (a) to show that $f_n, A_n, B_n \leq 5^n$ for all $n \geq 0$.

(c) Use the *Absolute Convergence* and *Comparison* Tests and only part (b) to show that the power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad A(x) = \sum_{n=0}^{\infty} A_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} B_n x^n,$$

converge if $|x| < 1/6$ (and thus determine smooth functions near $x=0$).

(d) Using only part (a), show that

$$f(x) = x + xf(x) + x^2 f(x), \quad A(x) = xA(x) + \frac{x}{(1-x)^2}, \quad B(x) = xB(x) + \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

Hint: You'll need to use identities such as the following:

$$\frac{1}{(1-x)^3} = \frac{1}{2}\left(\frac{1}{1-x}\right)'' = \frac{1}{2}\left(\sum_{n=0}^{\infty} x^n\right)'' = \frac{1}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2}.$$

(e) Using only part (d), express f_n, A_n, and B_n explicitly in terms of n.
Hint: use (d) to solve for f, A, and B and expand them into Taylor series around $x=0$ (partial fractions might help in the case of f); compare the result with the definitions of f, A, and B in (c).

Note: For f_n, you should end up with the formula in Problem G on PS6. There is a much simpler way of finding an explicit formula for A_n; so you can check your answer, but please deduce this formula from (b). The answer for B_n can be confirmed using induction (or google).