MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 10 (130pts)

WebAssign Problems 1 and 2 (7+13pts)

Determine the radius and interval of convergence of the following series:

$$(1)\quad \sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}\,, \qquad (2)\quad \sum_{n=1}^{\infty}\frac{3^n(x+4)^n}{\sqrt{n}}\,.$$

(1) We need to determine for which $x\neq 0$ the series converges. First apply the *Ratio Test* with $a_n = x^n/\sqrt{n}\neq 0$:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{|x|^{n+1}/\sqrt{n+1}}{|x|^n/\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1}}\cdot\frac{|x|^{n+1}}{|x|^n} = \sqrt{\frac{n}{n+1}}\cdot\frac{|x|^n\cdot|x|^1}{|x|^n} \\ &= \sqrt{\frac{n/n}{(n+1)/n}}\cdot|x| = \sqrt{\frac{1}{1+1/n}}\cdot|x| \longrightarrow \sqrt{\frac{1}{1+1/\infty}}\cdot|x| = |x|.\end{aligned}$$

Thus, the series converges if $|x|<1$ and diverges if $|x|>1$. So, the radius convergence is $R=1$, and the series converges at least for $x\in(-1,1)$. It remains to check what happens at the endpoints of this interval, i.e. whether each of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1^n}{\sqrt{n}}$$

converges or diverges. The latter series diverges by the p-Series Test with $p=1/2\le 1$. The former series is alternating, since the odd terms are negative, while the even terms are positive. Furthermore, $1/\sqrt{n}\longrightarrow 0$ as $n\longrightarrow\infty$ and $1/\sqrt{n+1}<1/\sqrt{n}$ for all n. Thus, this series converges by the *Alternating Series Test*. So, $x=-1$ is in the interval of convergence, while $x=1$ is not. Thus, the radius of convergence is $\boxed{1}$ and the interval of convergence is $\boxed{[-1,1)}$

(2) We need to determine for which $x\neq -4$ the series converges. First apply the *Ratio Test* with $a_n = 3^n(x+4)^n/\sqrt{n}\neq 0$:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{3^{n+1}|x+4|^{n+1}/\sqrt{n+1}}{3^n|x+4|^n/\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1}}\cdot\frac{3^{n+1}}{3^n}\cdot\frac{|x+4|^{n+1}}{|x+4|^n} = \sqrt{\frac{n}{n+1}}\cdot\frac{3^n\cdot 3^1}{3^n}\cdot\frac{|x+4|^n\cdot|x+4|}{|x+4|^n} \\ &= \sqrt{\frac{n/n}{(n+1)/n}}\cdot 3|x+4| = \sqrt{\frac{1}{1+1/n}}\cdot 3|x+4| \longrightarrow \sqrt{\frac{1}{1+1/\infty}}\cdot 3|x+4| = 3|x+4|.\end{aligned}$$

Thus, the series converges if $3|x+4|<1$ and diverges if $3|x+4|>1$. So, the radius convergence is $R=1/3$, and the series converges at least for $x\in(-13/3,-11/3)$. It remains to check what happens at the endpoints of this interval, i.e. whether each of the series

$$\sum_{n=1}^{\infty}\frac{3^n(-13/3+4)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{3^n(-11/3+4)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

converges or diverges. The latter series diverges by the p-Series Test with $p=1/2\le 1$. The former series is alternating, since the odd terms are negative, while the even terms are positive. Furthermore, $1/\sqrt{n}\longrightarrow 0$ as $n\longrightarrow\infty$ and $1/\sqrt{n+1}<1/\sqrt{n}$ for all n. Thus, this series converges by the *Alternating Series Test*. So, $x=-13/3$ is in the interval of convergence, while $x=-11/3$ is not. Thus, the radius of convergence is $\boxed{1/3}$ and the interval of convergence is $\boxed{[-13/3,-11/3)}$

WebAssign Problems 3-6 (6+5+5+9pts)

Find power series representations for the following functions and their intervals of convergence.

$$(3)\ f(x) = \frac{2}{3-x} \quad (4)\ f(x) = \frac{1}{x+10} \quad (5)\ f(x) = \frac{x}{2x^2+1} \quad (6)\ f(x) = \ln(5-x)$$

We will relate all four functions to the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,, \tag{1}$$

which converges if and only if $|x| < 1$.

(3) Factor 2 from the top and 3 from the bottom to make the fraction look like LHS of (1):

$$\frac{2}{3-x} = \frac{2\cdot 1}{3(1-x/3)} = \frac{2}{3}\cdot\frac{1}{1-(x/3)} = \frac{2}{3}\sum_{n=0}^{\infty}(x/3)^n = \frac{2}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n} = \boxed{\sum_{n=0}^{\infty}\frac{2x^n}{3^{n+1}}}$$

Since the series (1) converges if and only if $|x| < 1$ and we substitute $x/3$ for x, our series converges if and only if $|x/3| < 1$, i.e. for $|x| < 3$; so the interval of convergence is $\boxed{(-3,3)}$

(4) Switch the order of the terms and factor 10 from the denominator

$$\frac{1}{x+10} = \frac{1}{10(1+x/10)} = \frac{1}{10}\cdot\frac{1}{1-(-x/10)} = \frac{1}{10}\sum_{n=0}^{\infty}(-x/10)^n = \frac{1}{10}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{10^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{10^{n+1}}}$$

Since the series (1) converges if and only if $|x| < 1$ and we substitute $-x/10$ for x, our series converges if and only if $|-x/10| < 1$, i.e. for $|x| < 10$; so the interval of convergence is $\boxed{(-10,10)}$

(5) Switch the order of the terms in the denominator and factor x from the numerator:

$$\frac{x}{2x^2+1} = x\frac{1}{1-(-2x^2)} = x\sum_{n=0}^{\infty}(-2x^2)^n = x\sum_{n=0}^{\infty}(-2)^n x^{2n} = \boxed{\sum_{n=0}^{\infty}(-2)^n x^{2n+1}}$$

Since the series (1) converges if and only if $|x| < 1$ and we substitute $-2x^2$ for x, our series converges if and only if $|-2x^2| < 1$, i.e. for $x^2 < 1/2$; so the interval of convergence is $\boxed{(-1/\sqrt{2}, 1/\sqrt{2})}$

(6) Integrating the series (1) from $x = 0$, we obtain

$$\ln(1-x) = \int_0^x \frac{-1}{1-t}\mathrm{d}t = -\int_0^x \sum_{n=0}^{\infty} t^n \mathrm{d}t = -\sum_{n=0}^{\infty}\int_0^x t^n \mathrm{d}t = -\sum_{n=0}^{\infty}\frac{t^{n+1}}{n+1}\Big|_{t=0}^{t=x} = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty}\frac{x^n}{n}\,.$$

For the last equality, we can renumber the summands, replacing $n+1$ with n, so that the sum begins with $n = 0+1 = 1$ instead of $n = 0$.

We will relate the fourth function to the power series representation

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}\,. \tag{2}$$

By the *Ratio Test* (which is not effected by powers of n), the radius of convergence of this series is still $R=1$. This series diverges for $x=1$ by the *p-Series Test* and converges for $x=-1$ by the *Alternating Series Test*. So, the series (3) converges if and only if $-1 \le x < 1$.

In order to make our series look like LHS of (2), use log rules to pull out 5

$$\ln(5-x) = \ln\big(5\cdot(1-x/5)\big) = \ln 5 + \ln(1-x/5) = \ln 5 - \sum_{n=1}^{\infty}\frac{(x/5)^n}{n} = \boxed{\ln 5 - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}}$$

Since the series (2) converges if and only if $-1 \le x < 1$ and we substitute $x/5$ for x, our series converges if and only if $-1 \le x/5 < 1$, i.e. of $-5 \le x < 5$; so the interval of convergence is $\boxed{[-5,5)}$

WebAssign Problem 7 (4pts)

Evaluate the indefinite integral $\displaystyle\int \frac{t}{1-t^8}\mathrm{d}t$ *as a power series and find its radius of convergence.*

Put $x=t^8$ into series (1) and integrate:

$$\int \frac{t}{1-t^8}\mathrm{d}t = \int t\sum_{n=0}^{\infty}(t^8)^n\mathrm{d}t = \sum_{n=0}^{\infty}\int t^{8n+1}\mathrm{d}t = \boxed{C+\sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2}}$$

Since integration does not change the radius of convergence, the radius of convergence of the last series is the same as the radius of convergence of the series

$$\frac{1}{1-t^8} = \sum_{n=0}^{\infty} t^{8n}\,.$$

Since the series (1) converges if and only if $|x|<1$ and we substitute t^8 for x, the last series converges if and only if $|t^8|<1$, i.e. for $|t|<1$. Thus, the radius of convergence of the last series and of the integrated series is $\boxed{1}$ For $t=\pm 1$, the integrated series becomes

$$\sum_{n=0}^{\infty}\frac{(\pm 1)^{8n+2}}{8n+2} = \sum_{n=0}^{\infty}\frac{1}{8n+2}\,;$$

this series diverges by Limit Comparison the p-Series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ or Comparison with $\displaystyle\frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n}$. So the interval of convergence of the integrated series is still $(-1,1)$.

WebAssign Problem 8 (11pts)

(a) *Starting the geometric series* $f(x)=\displaystyle\sum_{n=0}^{\infty}x^n$, *find the sum of the series* $\displaystyle\sum_{n=1}^{\infty}nx^{n-1}$ *with* $|x|<1$.

Since $\displaystyle\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$ *if* $|x|<1$,

$$\sum_{n=1}^{\infty}nx^{n-1} = \left(\sum_{n=0}^{\infty}x^n\right)' = \left(\frac{1}{1-x}\right)' = \boxed{\frac{1}{(1-x)^2} \quad \textit{if } |x|<1}$$

(b) Find the sum of each of the following series:

$$\text{(i)}\quad \sum_{n=1}^{\infty} nx^n \quad |x|<1, \qquad \text{(ii)}\quad \sum_{n=1}^{\infty} \frac{n}{2^n}$$

By part (a),

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\cdot\frac{1}{(1-x)^2} = \boxed{\frac{x}{(1-x)^2}}$$

Since $|1/2|<1$,

$$\sum_{n=1}^{\infty}\frac{n}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} nx^n\Big|_{x=1/2} = \frac{x}{(1-x)^2}\Big|_{x=1/2} = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = \boxed{2}$$

(c) Find the sum of each of the following series:

$$\text{(i)}\quad \sum_{n=2}^{\infty} n(n-1)x^n \quad |x|<1, \qquad \text{(ii)}\quad \sum_{n=2}^{\infty}\frac{n^2-n}{2^n}, \qquad \text{(ii)}\quad \sum_{n=1}^{\infty}\frac{n^2}{2^n}.$$

Similarly to parts (a) and (b),

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\left(\sum_{n=0}^{\infty} x^n\right)'' = x^2\left(\frac{1}{1-x}\right)'' = x^2\frac{2}{(1-x)^3} = \boxed{\frac{2x^2}{(1-x)^3}}$$

Since $|1/2|<1$,

$$\sum_{n=2}^{\infty}\frac{n^2-n}{2^n} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} n(n-1)x^n\Big|_{x=1/2} = \frac{2x^2}{(1-x)^3}\Big|_{x=1/2} = \frac{2(1/2)^2}{(1-1/2)^3} = \frac{1/2}{1/8} = \boxed{4}$$

Combining this with (b) gives

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n} = \sum_{n=2}^{\infty}\frac{n^2-n}{2^n} + \sum_{n=1}^{\infty}\frac{n}{2^n} = 4+2 = \boxed{6}$$

Problem X.1 (5pts)

Determine the radius and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2x^n}{2\cdot 4\cdot\ldots\cdot 2n}$.

We need to determine for which $x\neq 0$ *the series converges. First apply the Ratio Test with* $a_n = n^2x^n/(2\cdot 4\cdot\ldots\cdot 2n)\neq 0$:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{(n+1)^2|x|^{n+1}/(2\cdot 4\cdot\ldots\cdot 2n\cdot 2(n+1))}{n^2|x|^n/(2\cdot 4\cdot\ldots\cdot 2n)} \\ &= \frac{(n+1)^2}{n^2}|x|\frac{2\cdot 4\cdot\ldots\cdot 2n}{2\cdot 4\cdot\ldots\cdot 2n\cdot(2n+2)} = \left(1+\frac{1}{n}\right)^2|x|\frac{1}{2n+2} \longrightarrow 0.\end{aligned}$$

So, the series converges for all x. *Thus, the radius of convergence is* $\boxed{\infty}$ *and the interval of convergence is* $\boxed{(-\infty,\infty)}$

Problem X.2 (5pts)

Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ *converges for* $x=-4$ *and diverges for* $x=6$*. Do the following series converge or diverge?*

$$(a)\ \sum_{n=0}^{\infty} c_n\,, \qquad (b)\ \sum_{n=0}^{\infty} c_n 8^n\,, \qquad (c)\ \sum_{n=0}^{\infty} c_n (-3)^n\,, \qquad (a)\ \sum_{n=0}^{\infty} (-1)^n c_n 9^n\,.$$

By the assumptions, the radius of convergence R *is at least* 4 *and at most* 6*. So the series converges if* $|x|<4$ *and diverges if* $|x|>6$*. Thus, the series* $\boxed{\text{(a),(c) converge}}$ *while* $\boxed{\text{(b),(d) diverge}}$
Note: the series (a) and (d) correspond to $x=1$ *and* $x=-9$*, respectively.*

Problem X.3 (5pts)

Find the interval of convergence of the series

$$f(x)=\sum_{n=0}^{\infty} c_n x^n, \qquad \textit{where} \quad c_n=\begin{cases} 1, & \textit{if } n \textit{ is even,}\\ 2, & \textit{if } n \textit{ is odd}\end{cases}$$

and an explicit formula for $f(x)$*.*

If $|x|\geq 1$*,* $c_n x^n$ *does not converge to zero and so the series* $\sum_{n=0}^{\infty} c_n x^n$ *diverges. If* $|x|<1$*,*

$$\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty} 1x^{2n}+\sum_{n=0}^{\infty} 2x^{2n+1}=\frac{1}{1-x^2}+\frac{2x}{1-x^2}=\frac{1+2x}{1-x^2}$$

The second equality holds because each of the two sums is a geometric series with $|r|=x^2<1$ *and thus converges. So the interval of convergence is* $\boxed{(-1,1)}$ *and* $\boxed{f(x)=\frac{1+2x}{1-x^2}}$
Note: it is **wrong** *to first split the sum into two geometric series and then claim that the original sum diverges for* $|x|\geq 1$ *because the two geometric series do.*

Problem X.4 (5pts)

Suppose the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ *is* R*. What is the radius of convergence of the power series* $\sum_{n=0}^{\infty} c_n x^{2n}$*?*

By assumption, the power series $\sum_{n=0}^{\infty} c_n x^n$ *converges if* $|x|<R$ *and diverges if* $|x|>R$*. Since* $x^{2n}=(x^2)^n$*, the power series* $\sum_{n=0}^{\infty} c_n x^{2n}$ *converges if* $|x^2|<R$ *and diverges if* $|x^2|>R$*; so it converges if* $|x|<\sqrt{R}$ *and diverges if* $|x|>\sqrt{R}$*. Thus, the radius of convergence of the second series is* $\boxed{\sqrt{R}}$

Problem X.5 (10pts)

Find power series representation for the function $f(x) = \left(\frac{x}{2-x}\right)^3$ *and its interval of convergence.*

We will relate this functions to the geometric series (1) which converges if and only if $|x| < 1$. *Differentiating the series (1) twice, we obtain*

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty}(x^n)' = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\left(\frac{1}{(1-x)^2}\right)' = \frac{1}{2}\left(\sum_{n=0}^{\infty}(n+1)x^n\right)' = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(x^n)' = \sum_{n=0}^{\infty}\frac{(n+1)n}{2}x^{n-1}$$
$$= \sum_{n=1}^{\infty}\frac{(n+1)n}{2}x^{n-1} = \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}x^n\,.$$

In both cases, we can first drop the $n=0$ *term because it is zero (second-to-last equality). We can then renumber the summands, replacing* $n-1$ *with* n, *so that the sum begins with* $n=1-1=0$ *instead of* $n=1$ *(last equality).*

We will relate the remaining function to the power series representation

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}x^n\,. \tag{3}$$

By the Ratio Test (which is not effected by powers of n*), the radius of convergence of this series is* $R=1$. *By the Test for Divergence, this series diverges for* $x=\pm1$ *(since the terms do not approach 0). So, the series (3) converges if and only if* $|x| < 1$.

Remark: More generally, the radius of convergence is not affected by differentiation/integration of power series, but the interval of convergence may be effected. However, differentiation can only drop both or either of the endpoints from the interval of convergence, but in this case there are no end-points to drop since the interval of convergence for the series (1) does not have any.

In order to make our series look like LHS of (3), pull out x *from the numerator and 2 from the denominator*

$$\left(\frac{x}{2-x}\right)^3 = \left(\frac{x}{2}\cdot\frac{1}{1-x/2}\right)^3 = \frac{x^3}{2^3}\cdot\left(\frac{1}{1-x/2}\right)^3 = \frac{x^3}{2^3}\sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}(x/2)^n = \frac{x^3}{2^3}\sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}\frac{x^n}{2^n}$$
$$= \boxed{\sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2^{n+4}}x^{n+3} = \sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{2^{n+1}}x^n}$$

Since the series (3) converges if and only if $|x| < 1$ *and we substitute* $x/2$ *for* x, *our series converges if and only if* $|x/2| < 1$, *i.e. for* $|x| < 2$; *so the interval of convergence is* $\boxed{(-2,2)}$

Remark: The answer to the first part of each of these questions must be a power series in x, *and not in* $(-x/10)$, $-2x^2$, *etc.. Thus, the expressions preceding the boxes cannot be the final answers.*

Problem X.6 (20pts)

Let $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. *Find the interval of convergence for* f, f', f''.

First find the radius of convergence for the f*-series using the Ratio Test with* $a_n = x^n/n^2$*:*

$$\frac{|a_{n+1}|}{|a_n|}=\frac{|x|^{n+1}/(n+1)^2}{|x|^n/n^2}=\frac{|x|^{n+1}}{|x|^n}\cdot\frac{n^2}{(n+1)^2}=|x|\cdot\frac{n^2/n^2}{(n+1)^2/n^2}=|x|\cdot\frac{1}{\big((n+1)/n\big)^2}$$
$$=|x|\cdot\frac{1}{(1+1/n)^2}\longrightarrow|x|\cdot 1.$$

So the f*-series converges if* $|x|<1$ *and diverges if* $|x|>1$*. Thus, the radius of convergence of the* f*-series is* 1*, and it remains to test convergence for* $|x|=1$*:*

$$f(1)=\sum_{n=1}^{\infty}\frac{1^n}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}\,,\qquad f(-1)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,.$$

The first series converges by the p-Series Test ($p=2>1$)*. The second series converges by the Absolute Convergence Test because*

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

is a convergent series (the Alternating Series Test can also be used for the $f(-1)$*-series). So the interval of convergence of the* f*-series is* $\boxed{[-1,1]}$

To find the f'*-series, just differentiate the* f*-series term by term:*

$$f'(x)=\sum_{n=1}^{\infty}n\frac{x^{n-1}}{n^2}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\,.$$

Since differentiation does not change the radius of convergence, the radius of convergence of this series is still 1*. However, differentiation may drop either or both of the end-points* ±1 *from the interval of convergence for the* f*-series. So we need to check convergence for* $x=\pm1$*:*

$$f'(1)=\sum_{n=1}^{\infty}\frac{1^{n-1}}{n}=\sum_{n=1}^{\infty}\frac{1}{n}\,,\qquad f'(-1)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,.$$

The first series diverges by the p-Series Test ($p=1\le1$)*. The second series converges by the Alternating Series Test because this series is alternating (odd terms are positive, even terms are negative),* $1/n\longrightarrow0$ *as* $n\longrightarrow\infty$*, and* $1/n>1/(n+1)$ *for all* n*. Thus, differentiation drops* $x=1$ *from the interval of convergence for the* f*-series, but not* $x=-1$*. So the interval of convergence of the* f'*-series is* $\boxed{[-1,1)}$

To find the f''*-series, just differentiate the* f'*-series term by term:*

$$f''(x)=\sum_{n=2}^{\infty}\frac{(n-1)x^{n-2}}{n}=\sum_{n=1}^{\infty}\frac{n}{n+1}x^{n-1}\,.$$

Since differentiation does not change the radius of convergence, the radius of convergence of this series is still 1*. However, differentiation may drop the remaining end-point $x=-1$ from the interval of convergence for the f'-series. So we need to check convergence for $x=-1$:*

$$f''(-1) = \sum_{n=1}^{\infty} \frac{n}{n+1}(-1)^n \,.$$

Since

$$\left|\frac{n}{n+1}(-1)^n\right| = \frac{n/n}{n/n+1/n} = \frac{1}{1+1/n} \longrightarrow 1,$$

the sequence $(-1)^n n/(n+1)$ does not converge to 0 and thus the series diverges (whether this sequence converges to anything nonzero or diverges, as is the case, is irrelevant for the convergence of the series). Thus, differentiation drops the only end-point $x=-1$ from the interval of convergence for the f'-series. So the interval of convergence of the f''-series is $\boxed{(-1,1)}$

Problem I (20pts)

(a; **7pts***) Show that the series*

$$g(z) = \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\right)$$

converges for every $z \neq m\pi$ for any nonzero integer m and that $g(0)=0$.

If $z \neq m\pi$ for any nonzero integer m, the series

$$g(z) = \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\right) = \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2\pi^2} = 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2} \tag{4}$$

converges because it looks like $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. By the Absolute Convergence Test, it is sufficient to check that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{|z^2-n^2\pi^2|}$ converges. Apply the Limit Comparison Test with $b_n=1/n^2>0$:

$$\frac{a_n}{b_n} = \frac{1/|z^2-n^2\pi^2|}{1/n^2} = \frac{n^2}{|z^2-n^2\pi^2|} = \frac{n^2/n^2}{|z^2-n^2\pi^2|/n^2} = \frac{1}{|z^2/n^2-\pi^2|} \longrightarrow \frac{1}{|z^2/\infty-\pi^2|} = \frac{1}{\pi^2}\,.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so does the series (4). By definition,

$$g(0) = \sum_{n=1}^{\infty}\left(\frac{1}{0-n\pi}+\frac{1}{0+n\pi}\right) = \sum_{n=1}^{\infty} 0 = 0.$$

Note: It is **wrong** *to split the sum given in the statement of the question into two because neither of the two resulting sums converges.*

(b; **12pts***) The function*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\right)$$

is thus well-defined for every $z \neq m\pi$ for any integer m. Show that

$$\lim_{z\longrightarrow 0} zf(z)=1, \qquad f(-z)=-f(z), \qquad f(z+\pi)=f(z), \qquad f(\pi/2)=0, \tag{5}$$

with the middle identities holding whenever either side is defined ($z \neq m\pi$ for any integer m).

Since the series (4) converges for all z close to 0 and depends continuously on z,

$$\lim_{z\longrightarrow 0} zf(z) = \lim_{z\longrightarrow 0}\big(1+zg(z)\big) = 1+0\cdot g(0)=1.$$

By (4),

$$f(-z)=\frac{1}{-z}+2(-z)\sum_{n=1}^{\infty}\frac{1}{(-z)^2-n^2\pi^2} = -\bigg(\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}\bigg) = -f(z).$$

For the third identity, look at partial sums:

$$\begin{aligned}
s_n(z+\pi) &= \frac{1}{z+\pi}+\sum_{k=1}^{k=n}\bigg(\frac{1}{z+\pi-k\pi}+\frac{1}{z+\pi+k\pi}\bigg)\\
&= \frac{1}{z+\pi}+\sum_{k=1}^{k=n}\frac{1}{z-(k-1)\pi}+\sum_{k=1}^{k=n}\frac{1}{z+(k+1)\pi}\\
&= \frac{1}{z+\pi}+\sum_{k=0}^{k=n-1}\frac{1}{z-k\pi}+\sum_{k=2}^{k=n+1}\frac{1}{z+k\pi} = \sum_{k=-n+1}^{k=n+1}\frac{1}{z+k\pi}\\
&= \sum_{k=-n}^{k=n}\frac{1}{z+k\pi}+\frac{1}{z+(n+1)\pi}-\frac{1}{z-n\pi} = s_n(z)+\frac{1}{z+(n+1)\pi}-\frac{1}{z-n\pi}\,.
\end{aligned}\tag{6}$$

Thus, taking the limit as $n\longrightarrow 0$,

$$f(z+\pi)=\lim_{n\longrightarrow\infty}s_n(z+\pi)=\lim_{n\longrightarrow\infty}s_n(z)=f(z).$$

The fourth identity follows from the second and third:

$$f(\pi/2)=f(\pi/2-\pi)=f(-\pi/2)=-f(\pi/2) \qquad\Longrightarrow\qquad f(\pi/2)=0.$$

Note: there is no issue with splitting the sum in (6) because it is a finite sum; no matter in what order you add up finitely many terms, their sum will be the same (it may change in an infinite sum).

*(c; **1pt**) What is the "simplest" function that satisfies all identities in (5)? (answer only)*

$$\boxed{\cot z=\frac{1}{\tan z}=\frac{\cos z}{\sin z}}$$

In fact,

$$\frac{\cos z}{\sin z}=f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\bigg(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\bigg). \tag{7}$$

Differentiating both sides of this identity with respect to z gives

$$\begin{aligned}
\frac{(-\sin z)(\sin z)-(\cos z)(\cos z)}{\sin^2 z} &= -\frac{1}{z^2}+\sum_{n=1}^{\infty}\bigg(-\frac{1}{(z-n\pi)^2}-\frac{1}{(z+n\pi)^2}\bigg)\\
\Longrightarrow \qquad \sum_{n=1}^{\infty}\bigg(\frac{1}{(z-n\pi)^2}+\frac{1}{(z+n\pi)^2}\bigg) &= \frac{1}{\sin^2 z}-\frac{1}{z^2}
\end{aligned}$$

Taking the limit as $z\longrightarrow 0$ of both sides gives

$$\begin{aligned}
\sum_{n=1}^{\infty}\bigg(\frac{1}{(n\pi)^2}+\frac{1}{(n\pi)^2}\bigg) &= \lim_{z\longrightarrow 0}\Bigg(\frac{1}{(z-\frac{z^3}{6}+\ldots)^2}-\frac{1}{z^2}\Bigg)=\lim_{z\longrightarrow 0}\Bigg(\frac{1}{z^2(1-\frac{z^2}{6}+\ldots)^2}-\frac{1}{z^2}\Bigg)\\
&= \lim_{z\longrightarrow 0}\frac{1}{z^2}\Bigg(\bigg(\sum_{n=0}^{\infty}\Big(\frac{z^2}{6}+\ldots\Big)^n\bigg)^2-1\Bigg)=\lim_{z\longrightarrow 0}\frac{1}{z^2}\Bigg(\Big(1+\frac{z^2}{3}+\ldots\Big)-1\Bigg)=\frac{1}{3}
\end{aligned}$$

on the last line $\ldots$ denotes terms involving z^4 and higher powers of z. From this, we obtain

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}}$$

From here, we can find the sum of the inverse squares of even integers,

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{24}\,,$$

as well as the sum of the inverse squares of odd integers:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}.$$

Taking further derivatives of (7) and then their limits at 0, you can compute $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{2p}}$ for any integer p; this will be a rational multiple of π^{2p}.

The identity (7) is equivalent to

$$h(z)\equiv\frac{f(z)}{\cot z}\ =\ 1.$$

The reason variable z was used in this problem, instead of x, is that everything above makes sense for a complex variable $z=x+\mathfrak{i}y$. The function $h(z)$ is holomorphic wherever it is defined (depends only on z, not $\bar{z}$, or x and y separately). By the last three identities in (5), it is defined everywhere and is periodic. One of the most important statements you'd learn in MAT 342 is that any such function is constant. In light of the first identity in (5), this constant is 1. This is the reason for (7).