MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 1 (90pts)

WebAssign Problem 1 (19pts)

(a) *For what values of* r *does the function* $f(x)=\mathrm{e}^{rx}$ *satisfy the differential equation*

$$2y''+y'-y=0\,?$$

Differentiate f and plug in into the equation:

$$f(x)=\mathrm{e}^{rx}, \qquad f'(x)=\mathrm{e}^{rx}\cdot r=r\cdot\mathrm{e}^{rx}, \qquad f''(x)=r\cdot\mathrm{e}^{rx}\cdot r=r^2\cdot\mathrm{e}^{rx}\,;$$
$$2f''+f'-f=2\cdot r^2\cdot\mathrm{e}^{rx}+r\cdot\mathrm{e}^{rx}-\mathrm{e}^{rx}=\big(2r^2+r-1\big)\mathrm{e}^{rx}=0.$$

Thus, f satisfies the differential equation if and only if

$$0=2r^2+r-1=(2r-1)(r+1);$$

so $r=1/2,-1$.

(b) *If* r_1 *and* r_2 *are the values of* r *you found in part (a), show that every member of the family of functions*

$$y(x)=a\,\mathrm{e}^{r_1x}+b\,\mathrm{e}^{r_2x}$$

is also a solution.

We need to show that the function

$$f(x)=a\,\mathrm{e}^{x/2}+b\,\mathrm{e}^{-x},$$

where a,b are any two fixed constants, solves the differential equation in part (a). So differentiate f and plug into the differential equation to compare the two sides:

$$f(x)=a\,\mathrm{e}^{x/2}+b\,\mathrm{e}^{-x}\,, \quad f'(x)=\frac{1}{2}a\,\mathrm{e}^{x/2}-b\,\mathrm{e}^{-x}\,, \quad f''(x)=\frac{1}{4}a\,\mathrm{e}^{x/2}+b\,\mathrm{e}^{-x}\,;$$
$$\begin{aligned}2f''+f'-f&=2\bigg(\frac{1}{4}a\mathrm{e}^{x/2}+b\mathrm{e}^{-x}\bigg)+\bigg(\frac{1}{2}a\mathrm{e}^{x/2}-b\mathrm{e}^{-x}\bigg)-\big(a\mathrm{e}^{x/2}+b\mathrm{e}^{-x}\big)\\&=\bigg(\frac{1}{2}+\frac{1}{2}-1\bigg)a\mathrm{e}^{x/2}+\big(2-1-1\big)b\mathrm{e}^{-x}\\&=0;\end{aligned}$$

so f indeed satisfies the differential equation.

WebAssign Problem 2 (16pts)

A function $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^4 - 6y^3 + 5y^2 \,.$$

(a) *What are the constant solutions of the equation?*

If the constant function $f(t) = C$ solves the equation, then

$$\begin{aligned} 0 = f'(t) &= f(t)^4 - 6f(t)^3 + 5f(t)^2 = C^4 - 6C^3 + 5C^2 \\ &= C^2\big(C^2 - 6C + 5\big) = C^2(C-1)(C-5); \end{aligned}$$

so $C = 0, 1, 5$.

(b) *For what values of y is y increasing?*

A solution y of the above differential equation is increasing if

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^4 - 6y^3 + 5y^2 > 0.$$

This is the case if

$$y^2(y^2 - 6y + 5) = y^2(y-1)(y-5) > 0,$$

i.e. if $y < 0$, $0 < y < 1$, or $y > 5$.

(c) *For what values of y is y decreasing?*

A solution y of the above differential equation is decreasing if

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^4 - 6y^3 + 5y^2 < 0.$$

This is the case if

$$y^2(y^2 - 6y + 5) = y^2(y-1)(y-5) < 0,$$

i.e. if $1 < y < 5$.

The conclusions in (a)-(c) can be restated using pictures as in Figure 1. The first diagram shows the y-axis and indicates the constant solutions by large dots. The arrows between the dots indicate whether the solutions in each interval go up and down. The second diagram shows the graphs of the constant solutions in thick lines. The arrows in each segment created by the constant solutions indicate whether the graphs of the other solutions in each segment rise or fall; these graphs must approach the two nearby constant solutions as $t \longrightarrow \pm\infty$. The first diagram makes sense only for *autonomous* first-order differential equations, i.e. equations of the form $y' = f(y)$, where y is a function of some variable t (note that there is no explicit dependence on t in the equation).

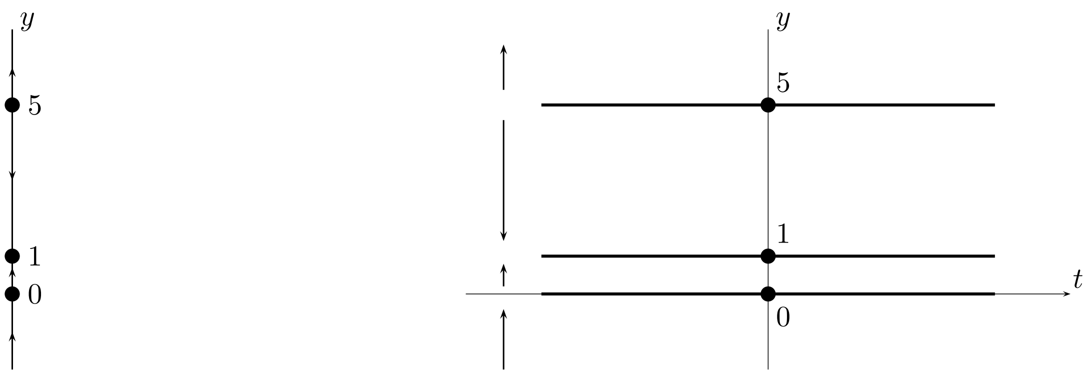

Figure 1: Diagram for WebAssign Problem 2

Problem 1.1 (10pts)

Verify that the function $f(t) = -t\cos t - t$ is a solution of the initial-value problem

$$t\,\frac{\mathrm{d}y}{\mathrm{d}t} = y + t^2\sin(t), \qquad y(\pi) = 0.$$

The function f is defined and continuously differentiable everywhere, as is every function in the differential equation, and

$$f(\pi) = -\pi\cos\pi - \pi = -\pi\cdot(-1) - \pi = 0 \quad \checkmark$$

thus, f satisfies the initial condition. To check that f satisfies the differential equation, differentiate f using the product rule and compare with the right-hand side of the equation plugging in f for y:

$$\begin{aligned}
f(t) &= -t\,\cos t - t;\\
\frac{\mathrm{d}f}{\mathrm{d}t} &= -\big(t'\cos t + t\cos' t\big) - t' = -(\cos t - t\sin t) - 1 = -\cos t + t\sin t - 1;\\
t\,\frac{\mathrm{d}f}{\mathrm{d}t} &= -t\cos t - t + t^2\sin t;\\
f(t) + t^2\sin(t) &= -t\cos t - t + t^2\sin(t);\\
\implies \qquad t\,\frac{\mathrm{d}f}{\mathrm{d}t} &= f + t^2\sin(t) \quad \checkmark
\end{aligned}$$

So f satisfies the differential equation. Since f satisfies the three conditions just checked, f is a solution of the initial-value problem.

Problem 1.2 (5pts)

The function with the given graph is a solution of one of the following differential equations:

$$\text{(A) } y' = 1 + xy, \qquad \text{(B) } y' = -2xy, \qquad \text{(C) } y' = 1 - 2xy.$$

Decide which is the correct equation and justify your answer.

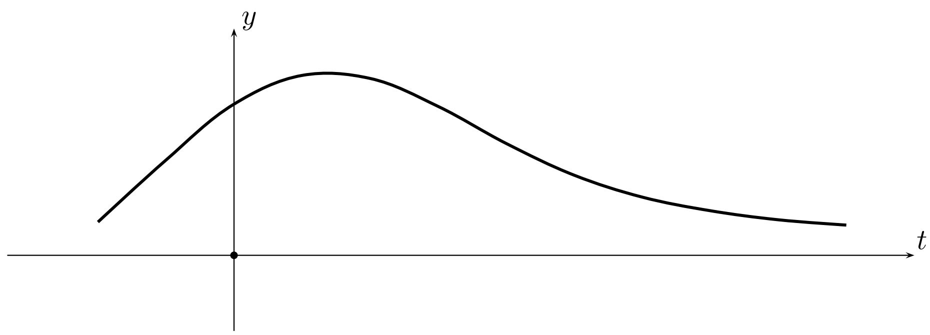

Figure 2: Graph for Problem 1.2

If y satisfied equation (A), $y'(x) = 1 + xy(x)$ would be positive when x and y are positive, so y would be increasing when x and y are positive; this is not the case for the pictured graph after the hump. If y satisfied equation (B), $y'(x) = -2xy(x)$ would be negative when x and y are positive, so y would be decreasing when x and y are positive; this is not the case for the pictured graph between the y-axis and the hump. So the answer must be (C).

Problem A (40pts)

(a; **4pts**) *State the two Fundamental Theorems of Calculus (answer only).*

(a-i) If F is a continuously differentiable function on the interval (a, b) and $t_0 \in (a, b)$, then

$$F(t) = F(t_0) + \int_{t_0}^{t} F'(s)ds \qquad \text{for all } \ t \in (a, b). \tag{1}$$

(a-ii) If f is a continuous function on the interval (a, b), $t_0 \in (a, b)$,

$$\text{and} \qquad F(t) \equiv \int_{t_0}^{t} f(s)ds \qquad \text{for all } \ t \in (a, b),$$
$$\text{then} \qquad F'(t) = f(t) \qquad \text{for all } \ t \in (a, b).$$

(b; **2pts**) *State the chain rule for one-variable differentiation (answer only).*

If f and g are continuously differentiable functions on (a,b) and (c,d), respectively, and $a<g(t)<b$ for all $t\in(c,d)$, then the function

$$h(t) \equiv f(g(t)), \qquad t \in (c,d),$$

is defined and continuously differentiable on (c,d) and

$$h'(t) = f'(g(t)) \cdot g'(t) \qquad \text{for all} \;\; t \in (c,d).$$

(c; **2pts**) *State the product rule for one-variable differentiation (answer only).*

If f and g are continuously differentiable functions on (a,b), then the function

$$h(t) \equiv f(t) \cdot g(t), \qquad t \in (a,b),$$

is also continuously differentiable and

$$h'(t) = f'(t) \cdot g(t) + f(t) \cdot g'(t) \qquad \text{for all} \;\; t \in (a,b). \tag{2}$$

(d; **2pts**) *If a is a real number and $f(x)=x^a$, what is $f'(x)$? (answer only)* $\boxed{f'(x) = a \cdot x^{a-1}}$

(e; **2pts**) *If $f(x)=e^x$, what is $f'(x)$? (answer only)* $\boxed{f'(x) = e^x}$

(f; **6pts**) *State the quotient rule for one-variable differentiation. Deduce it from (b)-(d).*

If f and g are continuously differentiable functions on (a,b) and $g(t)\neq 0$ for all $t\in(a,b)$, then the function

$$h(t) \equiv f(t)/g(t), \qquad t \in (a,b),$$

is also continuously differentiable and

$$h'(t) = \frac{f'(t) \cdot g(t) - f(t) \cdot g'(t)}{g(t)^2} \qquad \text{for all} \;\; t \in (a,b). \tag{3}$$

In order to prove (3), we apply (c) to the functions f and $G(t)=1/g(t)$. Since $h=f\cdot G$,

$$h'(t) = f'(t) \cdot G(t) + f(t) \cdot G'(t) = \frac{f'(t)}{g(t)} + f(t) \cdot G'(t). \tag{4}$$

In order to compute $G'(t)$, we apply (b) to the functions $y(u)=u^{-1}$ and g and use (d) with $a=-1$. Since $G(t)=y(g(t))$,

$$G'(t) = y'(g(t)) \cdot g'(t) = (-1) \cdot g(t)^{-2} \cdot g'(t) = -\frac{g'(t)}{g(t)^2}. \tag{5}$$

The quotient rule, i.e. (3), is obtained by plugging (5) into (4).

(g; **6pts**) *State the change-of-variables formula for one-variable integration. Deduce it from (a) and (b).*

If f is a continuous function on (a,b), g is a continuously differentiable function on (c,d) such that $a<g(t)<b$ for all $t\in(c,d)$, $t_0\in(c,d)$,

$$\text{and} \qquad F(x) \equiv \int_{g(t_0)}^{x} f(y)dy \qquad \text{for all} \;\; x \in (a,b), \tag{6}$$

$$\text{then} \qquad \int_{t_0}^{t} f(g(s)) \cdot g'(s)ds = F(g(t)) \qquad \text{for all} \;\; t \in (c,d). \tag{7}$$

By (b) applied to F, g, and $h(t)=F(g(t))$,

$$h'(t) = F'(g(t)) \cdot g'(t). \tag{8}$$

Since $h(t_0)=0$ by (6), the change-of-variables formula, i.e. (7), follows from (1) and (8).

(h; **6pts**) *State the integration-by-parts formula for one-variable integration. Deduce it from (a) and (c).*

If f and g are continuously differentiable functions on (a,b) and $t_0\in(a,b)$,

$$\int_{t_0}^{t} f(s) \cdot g'(s)ds = \big(f(t)g(t) - f(t_0)g(t_0)\big) - \int_{t_0}^{t} f'(s) \cdot g(s)ds \qquad \text{for all} \;\; t \in (a,b). \tag{9}$$

Rearranging (2) with $h(s)=f(s)g(s)$, we obtain

$$f(s) \cdot g'(s) = h'(s) - f'(s) \cdot g(s) \qquad \text{for all} \;\; s \in (a,b). \tag{10}$$

The integration-by-parts formula, i.e. (9), is obtained by integrating both sides of (10) and applying (1) to the middle term.

(i; **10pts**) *Suppose $a=a(t)$ is a smooth function, c is a real number,*

$$f(t) = \int_{c}^{t} a(s)ds, \qquad \textit{and} \qquad h(t)=e^{f(t)}.$$

Using (a), (b), and (e), show that the function $h=h(t)$ satisfies the differential equation $h'=ah$.

Compute $h'(t)$ and compare with $a(t)h(t)$. In order to compute $h'(t)$, apply (b) to the functions $F(u)=e^u$ and $G(t)=f(t)$. Since $h(t)=F(G(t))$,

$$h'(t) = F'(G(t)) \cdot G'(t) = e^{G(t)} \cdot G'(t) = e^{f(t)} \cdot f'(t) = f'(t) \cdot e^{f(t)}. \tag{11}$$

The second equality in (11) is a consequence of (e). Since $f'(t)=a(t)$ by (a-ii),

$$h'(t) = a(t) \cdot e^{f(t)} = a(t)h(t).$$

So, the function $h=h(t)$ satisfies the differential equation $h'=ah$.

Note: There are a number of ways of phrasing (a)-(c) and (f)-(h).