MAT 127: Calculus C, Fall 2010
Solutions to Final Exam

Problem 1 (10pts)

Determine whether each of the following sequences or series converges or not. In each case, clearly circle either **YES** *or* **NO**, *but not both. Each correct answer is worth 2 points.*

(a) the sequence $a_n = 1 + \dfrac{\cos^3 n}{n}$ **YES** **NO**

Since $\cos^3(n)/n \longrightarrow 0$ (because $|\cos^3(n)| \le 1$), $a_n \longrightarrow 1$

(b) the sequence $a_n = n^2(1 - e^{1/n})$ **YES** **NO**

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} \frac{1-\mathrm{e}^{1/n}}{(1/n)^2} = \lim_{x\longrightarrow 0+} \frac{1-\mathrm{e}^x}{x^2} = \lim_{x\longrightarrow 0+} \frac{0-\mathrm{e}^x}{2x} = -\frac{1}{2}\lim_{x\longrightarrow 0+}\frac{1}{x} = -\infty\,.$$

The third equality uses l'Hospital, which is applicable here because $(1-\mathrm{e}^x), x^2 \longrightarrow 0$ as $x \longrightarrow 0$.

Alternatively, using $\mathrm{e}^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, we find that

$$a_n = n^2\big(1 - (1 + \frac{1}{1!n} + \frac{1}{2!n^2} + \frac{1}{3!n^3} + \ldots)\big) = -n - \frac{1}{2} - \frac{1}{6n} - \ldots$$

where $\ldots$ involve $1/n^2$, $1/n^3$, and so on. As $n \longrightarrow \infty$, a_n thus approaches $-\infty$ and so diverges.

(c) the series $\sum\limits_{n=1}^{\infty} \dfrac{n+(-1)^n}{n^2+1}$ **YES** **NO**

$\dfrac{n+(-1)^n}{n^2+1}$ looks like $n/n^2 = 1/n$: $\quad \dfrac{(n+(-1)^n)/(n^2+1)}{1/n} = \dfrac{n^2+(-1)^n n}{n^2+1} = \dfrac{1+(-1)^n/n}{1+1/n^2} \longrightarrow 1$

Since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges by the p-series test ($p{=}1 \le 1$) and both series are nonnegative, by the Limit Comparison Test our series also diverges.

Alternatively, $\quad \sum\limits_{n=1}^{\infty} \dfrac{n+(-1)^n n}{n^2+1} = \sum\limits_{n=1}^{\infty} \dfrac{n}{n^2+1} + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2+1}$

The last series on RHS converges by the Alternating Series Test (it is alternating and approaching 0, and the absolute values of its terms are decreasing). The first series on RHS diverges by Limit Comparison to the p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, Comparison to $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2+n^2} = \dfrac{1}{2}\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, or by the Integral Test with $f(x) = x/(x^2+1)$. Since the sum of a divergent series and a *convergent* series is divergent, our series diverges.

(d) the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n}{2n+1}$ **YES** **NO**

Since the sequence $(-1)^n \dfrac{n}{2n+1} = (-1)^n \dfrac{1}{2+1/n} \longrightarrow \pm\dfrac{1}{2}$ does not approach 0, our series diverges by the Test for Divergence.

(e) the series $\sum\limits_{n=1}^{\infty} \dfrac{2^n}{\sqrt{3^n+5^n}}$ **YES** **NO**

$\dfrac{2^n}{\sqrt{3^n+5^n}}$ looks like $\dfrac{2^n}{\sqrt{5^n}} = \left(\dfrac{2}{\sqrt{5}}\right)^n$: $\quad \dfrac{2^n/\sqrt{3^n+5^n}}{(2/\sqrt{5})^n} = \dfrac{1}{\sqrt{3^n+5^n}/\sqrt{5^n}} = \dfrac{1}{\sqrt{(3/5)^n+(5/5)^n}} \longrightarrow 1$

Since $2 < \sqrt{5}$, the geometric series $\sum\limits_{n=1}^{\infty} \left(\dfrac{2}{\sqrt{5}}\right)^n$ converges. Since both series are nonnegative, by the Limit Comparison Test our series also converges.

Problem 2 (10pts)

Answer Only. *Put your answer to each question in the corresponding box in the simplest possible form. No credit will be awarded if the answer in the box is wrong; partial credit may be awarded if the answer in the box is correct, but not in the simplest possible form.*

(a; 5pts) *Write the number* $1.1\overline{09} = 1.1090909\ldots$ *as a simple fraction*

$\boxed{\dfrac{61}{55}}$

$$\begin{aligned} 1.1\overline{09} &= 1.1 + .009 + .009 \cdot \frac{1}{100} + .009 \cdot \frac{1}{100^2} + \ldots \\ &= \frac{11}{10} + \frac{9/1000}{1-\frac{1}{100}} = \frac{11}{10} + \frac{9/10}{99} = \frac{11}{10} + \frac{1}{110} = \frac{121+1}{110} = \frac{61}{55} \end{aligned}$$

Grading: wrong answer 0pts; as above 5pts; $1\frac{6}{55}$ or fraction not simplified 4pts; both issues 3pts

(b; 5pts) *Find the limit of the sequence recursively defined by*

$\boxed{3}$

$$a_1 = 4, \qquad a_{n+1} = 4 - \frac{3}{a_n} \quad n \geq 1.$$

Assume that this sequence converges.

If $a = \lim a_n$, $a = 4 - 3/a$, or $a^2 - 4a + 3 = 0$, or $(a-1)(a-3) = 0$. Thus, either $a = 1$ or $a = 3$. On the other hand, $a_n \geq 3$ for all n. This is true for $n{=}1$ (since $a_1 = 4$). If $a_n \geq 3$ for some n, then

$$a_{n+1} = 4 - \frac{3}{a_n} \geq 4 - \frac{3}{3} = 3.$$

So, by induction $a_n \geq 3$ for all n. Thus, the limit of this sequence must be at least 3 as well.

Grading: answer as above 5pts; 0pts otherwise

Note: In this case, it is possible to guess the answer by writing out the first few terms of the sequence:

$$\begin{aligned} &a_1 = 4, \qquad a_2 = 4 - \frac{3}{4} = \frac{13}{4} = 3\frac{1}{4}, \qquad a_3 = 4 - \frac{3}{13/4} = \frac{52-12}{13} = \frac{40}{13} = 3\frac{1}{13}, \\ &a_4 = 4 - \frac{3}{40/13} = \frac{160-39}{40} = \frac{121}{40} = 3\frac{1}{40}. \end{aligned}$$

This sequence seems to be approaching 3.

Problem 3 (20pts)

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence.

(a; 10pts) $f(x) = x^2 + 2x$ around $x = -2$

In this case, all derivatives can be computed:

$$f^{\langle 0\rangle}(x) = x^2 + 2x \implies f^{\langle 0\rangle}(-2) = 0, \qquad f^{\langle 1\rangle}(x) = 2x + 2 \implies f^{\langle 1\rangle}(-2) = -2,$$
$$f^{\langle 2\rangle}(x) = 2 \implies f^{\langle 2\rangle}(-2) = 2,$$

and $f^{\langle n\rangle}(x) = 0$ if $n \geq 3$. So by the Main Taylor Formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(-2)}{n!}(x+2)^n = \frac{0}{0!}x^0 + \frac{-2}{1!}(x+2)^1 + \frac{2}{2!}(x+2)^2$$
$$= \boxed{-2(x+2) + (x+2)^2}$$

Since this series is a sum of finitely many (two) terms, it converges for all x. So the interval of convergence is $\boxed{(-\infty, \infty)}$, while the radius is $\boxed{\infty}$

Remark: you can check the Taylor series expansion by expanding the expression in the long box above and getting $x^2 + 2x$.

Grading: statement of general Taylor formula 1pt, with $a = -2$ 3pts (*not* in addition to 1pt); vanishing of higher derivatives 1pt and computation of the remaining derivatives 2pts (no separate statement is required if general Taylor formula with $a = -2$ is stated and correctly applied in this case); final answer 1pt; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct); at least 2pts off if final answer is not a polynomial in $(x+2)$.

(b; 10pts) $f(x) = \dfrac{x}{4 + x^2}$ around $x = 0$

Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and this power series converges if $|x| < 1$,

$$\frac{x}{4+x^2} = \frac{x}{4} \cdot \frac{1}{1-(-x^2/4)} = \frac{x}{4}\sum_{n=0}^{\infty}\left(-\frac{x^2}{4}\right)^n = \frac{x}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}(x^2)^n = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n+1}}$$

and this series converges whenever

$$|x^2/4| < 1 \iff x^2 < 4 \iff -2 < x < 2;$$

so the interval of convergence is $\boxed{(-2, 2)}$ and the radius is $\boxed{2}$

Grading: use of correct standard power series 2pts (no separate statement is required if properly used in the given case); substitution and multiplication statement 1pt each; 3pts for simplifying to a power series in x; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct, except end-points error 1pt off).

(c; bonus 10pts) $\quad f(x) = \dfrac{1}{5 - 12x^2 + 4x^4}$ around $x = 0$

First, factor out the denominator:

$$\frac{1}{4x^4 - 12x^2 + 5} = \frac{1}{(2x^2-1)(2x^2-5)} = \frac{1}{(1-2x^2)} \cdot \frac{1}{(5-2x^2)}$$

Since $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ if $|x| < 1$,

$$\frac{1}{1-2x^2} = \sum_{n=0}^{\infty} (2x^2)^n = \sum_{n=0}^{\infty} 2^n x^{2n} \qquad \text{if } |2x^2| < 1$$

$$\frac{1}{5-2x^2} = \frac{1/5}{1-2x^2/5} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{2x^2}{5}\right)^n = \frac{1}{5}\sum_{n=0}^{\infty}\frac{2^n}{5^n}(x^2)^n = \sum_{n=0}^{\infty}\frac{2^n}{5^{n+1}}x^{2n} \qquad \text{if } |2x^2/5| < 1.$$

Using partial fractions thus gives

$$\frac{1}{4x^4-12x^2+5} = \frac{1}{(2x^2-1)(2x^2-5)} = \frac{1}{(-5)-(-1)}\left(\frac{1}{2x^2-1} - \frac{1}{2x^2-5}\right) = \frac{1}{-4}\left(\frac{1}{2x^2-1} - \frac{1}{2x^2-5}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1-2x^2} - \frac{1}{5-2x^2}\right) = \frac{1}{4}\left(\sum_{n=0}^{\infty} 2^n x^{2n} - \sum_{n=0}^{\infty}\frac{2^n}{5^{n+1}}x^{2n}\right) = \boxed{\sum_{n=0}^{\infty} 2^{n-2}\left(1 - \frac{1}{5^{n+1}}\right)x^{2n}}$$

Since the $1/(1-2x^2)$ series converges whenever $x^2 < 1/2$, while the $1/(5-2x^2)$ series converges whenever $x^2 < 5/2$, the difference converges whenever $x^2 < 1/2$. So the interval of convergence is $\boxed{(-1/\sqrt{2}, 1/\sqrt{2})}$ and the radius is $\boxed{1/\sqrt{2}}$

Alternatively, multiplication of power series can be used:

$$\frac{1}{5 - 12x^2 + 4x^4} = \frac{1}{(1-2x^2)} \cdot \frac{1}{(5-2x^2)} = \left(\sum_{n=0}^{\infty} 2^n x^{2n}\right) \cdot \frac{1}{5}\left(\sum_{n=0}^{\infty}\frac{2^n}{5^n}x^{2n}\right)$$

$$= \frac{1}{5}\left(1 + 2x^2 + 2^2(x^2)^2 + \dots\right)\left(1 + \frac{2}{5}x^2 + \frac{2^2}{5^2}(x^2)^2 + \dots\right)$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left(1\cdot\left(\frac{1}{5}\right)^n + 1\cdot\left(\frac{1}{5}\right)^{n-1} + \dots + 1\cdot\left(\frac{1}{5}\right) + 1\cdot 1\right)2^n x^{2n}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{5}\right)^{n+1} - 1^{n+1}}{\frac{1}{5} - 1}2^n x^{2n} = \boxed{\sum_{n=0}^{\infty} 2^{n-2}\left(1 - \frac{1}{5^{n+1}}\right)x^{2n}}$$

Since the last power series is the difference of a power series convergent for $x^2 < 1/2$ and a power series convergent for $x^2 < 5/2$, it converges for $x^2 < 1/2$.

Grading: product decomposition of the fraction 1pt; power series for $1/(1–2x^2)$ 1pt; power series for $1/(5–2x^2)$ 2pts; 3pts for either the partial fraction or multiplication of power series computation to the answer; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct, except end-points error 1pt off).

Problem 4 (20pts)

(a; 8pts) *Find the radius and interval of convergence of the power series*

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

To find the radius of convergence, use the Ratio Test with $a_n = x^n/\sqrt{n} \neq 0$:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}/\sqrt{n+1}}{|x|^n/\sqrt{n}} = |x|\sqrt{\frac{n}{n+1}} = \sqrt{\frac{1}{1+1/n}}\,|x| \longrightarrow \sqrt{1+0}\,|x| = |x|.$$

So the series converges if $|x|<1$ and diverges if $|x|>1$. Thus, the radius of convergence is $\boxed{1}$ and it remains to check convergence for $x=\pm 1$, i.e. whether each of the series

$$f(1) = \sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad \text{and} \qquad f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges. The first series diverges by the p-Series Test with $p = 1/2$. The second series converges by the Alternating Series Test, since it is alternating (odd terms are negative, even terms are positive), decreasing in absolute value ($1/\sqrt{n} > 1\sqrt{n+1}$), and approaching 0 ($1/\sqrt{n} \longrightarrow 0$). So, the right end-point is not in the interval of convergence, while the left one is; thus the interval of convergence is $\boxed{[-1,1)}$

Grading: radius of convergence 1pt, justification 3pts; convergence/divergence at the end-points and justification 3pts; interval of convergence 1pt; at least 3pts off if the radius is found to be ∞

(b; 4pts) *Find* $\lim\limits_{x\longrightarrow 0} \frac{f(x)-x}{x^2}$

$$\frac{f(x)-x}{x^2} = \frac{(x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \ldots) - x}{x^2} = \frac{\frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \ldots}{x^2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}x + \ldots \underset{x\longrightarrow 0}{\longrightarrow} \frac{1}{\sqrt{2}} + 0 = \boxed{\frac{1}{\sqrt{2}}}$$

Grading: expanding the series 2pts; indication of computation to the answer 2pts; use of l'Hospital rule (twice) is fine, but the 0/0 assumption must be checked each time, with 1pt off for each of the times the check is missing; if the answer is correct, at least 1pt for this part.

(c; 8pts) *Find the Taylor series expansion for the function* $g = g(x)$ given by

$$g(x) = \int_0^x \frac{f(u) - u}{u^2} \mathrm{d}u$$

around $x=0$. *What are the radius and interval of convergence of this power series?*

Since $\dfrac{f(u) - u}{u^2} = u^{-2} \sum_{n=2}^{\infty} \dfrac{u^n}{\sqrt{n}} = \sum_{n=2}^{\infty} \dfrac{u^{n-2}}{\sqrt{n}}$,

$$g(x) = \int_0^x \frac{f(u) - u}{u^2} \mathrm{d}u = \sum_{n=2}^{\infty} \frac{u^{n-1}}{\sqrt{n}(n-1)} \Big|_{u=0}^{u=x} = \boxed{\sum_{n=2}^{\infty} \frac{x^{n-1}}{(n-1)\sqrt{n}} = \sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n+1}}}$$

Since integration does not change the radius of convergence of a power series, the radius of convergence of this power series is still $\boxed{1}$ Since integration can only add end-points to the interval of convergence, this series converges at $x = -1$ by part (a) and we we need to check convergence of the series

$$g(1) = \sum_{n=1}^{\infty} \frac{1^n}{n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$$

Since $1/n\sqrt{n+1}$ looks like $1/n\sqrt{n} = 1/n^{3/2}$,

$$\frac{\sqrt{1}/n\sqrt{n+1}}{1/n^{3/2}} = \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}} = \sqrt{\frac{1}{1+1/n}} \longrightarrow 1,$$

$\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges by the p-Series test ($p = 3/2 > 1$), and both series are nonnegative, the $g(1)$ series also converges. So the end-point $x = 1$ is added to the interval of convergence in (a); thus the interval of convergence is $\boxed{[-1, 1]}$

Grading: integrand as a power series 2pts; power series for g 1pt; radius of convergence and explanation 1pt each (use of ratio test is ok); convergence/divergence at each of the end-points and justification 3pts (either *by Limit Comparison Test* or statement of the 3 assumptions suffices; limit computation not required; Comparison Test can also be used); no penalty for carry-over errors from (a) or inconsistencies in *answers* between (a) and (c).

Problem 5 (20pts)

Show that the following series are convergent and find their sums.

(a; 10pts) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{9^n(2n)!}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{9^n(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\bigg|_{x=\frac{\pi}{3}}.$$

Since the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ converges for all x and its sum equals $\cos x$, its evaluation at $x=\pi/3$, i.e. the infinite series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{9^n(2n)!}$, converges and equals $\cos(\pi/3)=\boxed{1/2}$

You can also justify convergence using the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|}=\frac{\pi^{2(n+1)}/(9^{n+1}(2(n+1))!)}{\pi^{2n}/(9^n(2n)!)}=\frac{\pi^{2n+2}}{\pi^{2n}}\cdot\frac{9^n}{9^{n+1}}\cdot\frac{(2n)!}{(2n+2)!}=\pi^2\cdot\frac{1}{9}\cdot\frac{1}{(2n+1)(2n+2)}\longrightarrow 0.$$

Since $0<1$, the series converges by the Ratio Test.

Grading: correct power series 3pts; evaluation point 3pts; sum of power series, $\cos(x)$, 1pt; answer 1pt; justification of convergence 2pts; if *only* the convergence part is done, up to 5pts for part (a).

(b; 10pts) $\displaystyle\sum_{n=1}^{\infty}\frac{n2^n}{3^n}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty}\frac{n2^n}{3^n}=\sum_{n=1}^{\infty}n\left(\frac{2}{3}\right)^n=\sum_{n=1}^{\infty}nx^n\bigg|_{x=2/3}.$$

Now use standard power series to sum up the power series:

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}\ \text{if } |x|<1\implies\sum_{n=1}^{\infty}nx^n=x\sum_{n=1}^{\infty}nx^{n-1}=x\left(\sum_{n=0}^{\infty}x^n\right)'=x\left(\frac{1}{1-x}\right)'=\frac{x}{(1-x)^2}\ \text{if } |x|<1.$$

Since the power series $\displaystyle\sum_{n=1}^{\infty}nx^n$ converges whenever $|x|<1$ and its sum (in those cases) is $\dfrac{x}{(1-x)^2}$, its evaluation at $x=\dfrac{2}{3}$, i.e. the infinite series $\displaystyle\sum_{n=1}^{\infty}\frac{n2^n}{3^n}$, converges and equals $\dfrac{2/3}{(1-2/3)^2}=\dfrac{2/3}{1/9}=\boxed{6}$

You can also justify convergence using the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|}=\frac{(n+1)2^{n+1}/3^{n+1}}{n2^n/3^n}=\frac{n+1}{n}\cdot\frac{2^{n+1}}{2^n}\cdot\frac{3^n}{3^{n+1}}=\left(1+\frac{1}{n}\right)\cdot 2\cdot\frac{1}{3}\longrightarrow(1+0)\cdot\frac{2}{3}=\frac{2}{3}.$$

Since $2/3<1$, the series converges by the Ratio Test.

Grading: correct power series and evaluation point 2pts each; sum of power series, $x/(1-x)^2$, and justification 1pt each; rest of computation 2pts; mention of range of convergence for the power series and justification of convergence of infinite series 1pt each (if no mention of interval of convergence is made, but convergence is justified directly, via RT, still 2pts); if *only* the convergence part is done, up to 5pts for part (b).

Problem 6 (10pts)

All questions in this problem refer to the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+2)\cdot n!\cdot 2^n}$$

(a; 3pts) *Explain why this series converges.*

This series converges because it is alternating (odd terms are positive, even terms are negative),

$$\frac{1}{(n+2)n!2^n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow 0, \qquad \text{and} \qquad \frac{1}{(n+2)n!2^n} > \frac{1}{((n+1)+2)(n+1)!2^{n+1}}.$$

It also converges by the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/(((n+1)+2)(n+1)!2^{n+1})}{1/((n+2)n!2^n)} = \frac{n+2}{n+3}\cdot\frac{n!}{(n+1)!}\cdot\frac{2^n}{2^{n+1}} = \frac{1+2/n}{1+3/n}\cdot\frac{1}{n+1}\cdot\frac{1}{2} \longrightarrow 1\cdot 0\cdot\frac{1}{2} = 0.$$

Since $0<1$, the series converges by the Ratio Test.

Grading: assumption checks are required for full credit, names of tests are not

(b; 4pts) *What is the minimal number of terms required to approximate the sum of this series with error less than* $1/1000$*? Justify your answer.*

Since the 3 assumptions for the Alternating Series Test hold,

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+2)n!2^n} - \sum_{n=1}^{n=m}\frac{(-1)^{n-1}}{(n+2)n!2^n}\right| = \left|\sum_{n=m+1}^{\infty}\frac{(-1)^{n-1}}{(n+2)n!2^n}\right| < |a_{m+1}| = \frac{1}{((m+1)+2)(m+1)!2^{m+1}}.$$

We need to choose the smallest m so that $(m+3)(m+1)!2^{m+1} \ge 1000$. Plugging in $m = 1,2,3$, we find that the smallest value that works is $\boxed{3}$

Note: According to the book's recipe, you need to take $m=3$ as done above because this is the smallest value of m for which the book's upper-bound on the remainder of the infinite series is no greater than the required precision (with $m=2$, the upper-bound is $1/(5\cdot6\cdot8)=1/240 > 1/1000$). However, the remainder is smaller than the upper bound, so that a smaller m could still work. In fact, $m=2$ would not work in this case because a lower bound for the remainder for A.S.T. is given by the sum of the first *two* dropped terms; in this case, this bound is

$$\frac{1}{((m+1)+2)(m+1)!2^{m+1}} - \frac{1}{((m+2)+2)(m+2)!2^{m+2}} = |a_{m+1} + a_{m+2}| < \left|\sum_{n=m+1}^{\infty}\frac{(-1)^{n-1}}{(n+2)n!2^n}\right|.$$

If $m=2$, this lower bound is $1/240 - 1/(6\cdot24\cdot16) = 43/(720\cdot16) > 1/1000$.

Grading: *Alternating Series Estimation Theorem* statement for the given case 2pts; conclusion that $m=3$ 2pts; bonus 5pts (all or nothing) for full justification that $m=3$ is actually sharp (as in *Note* above) (c; 3pts) *Based on your answer in part (b), estimate the sum of this series with error less than* $1/1000$*; leave your answer as a simple fraction* p/q *for some integers* p *and* q *with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why. (If you do not know how to do (b), take the answer to (b) to be 2).*

Based on part (b), the required estimate is

$$\sum_{n=1}^{n=m}\frac{(-1)^{n-1}}{(n+2)n!2^n}=\sum_{n=1}^{n=3}\frac{(-1)^{n-1}}{(n+2)n!2^n}=\frac{(-1)^0}{3\cdot 1!\cdot 2}+\frac{(-1)^1}{4\cdot 2!\cdot 4}+\frac{(-1)^2}{5\cdot 3!\cdot 8}=\frac{5\cdot 8\cdot 2-5\cdot 3+2}{5\cdot 6\cdot 8\cdot 2}=\boxed{\frac{67}{480}}$$

This is an $\boxed{\text{over-estimate}}$ for the infinite sum, because the last term used is positive.

Grading: computation of finite sum 2pts (1pt off if fraction not simplified); under/over-estimate (depending on m) with justification 1pt; if using $m=2$, only the first two terms should be taken, resulting in $13/96$, which is an an under-estimate because the last term used is negative.

(d; bonus 8pts) *Find the sum of the infinite series exactly.*

We need to relate this infinite series to some power series evaluated at some point. Because of $n!$ in the denominator, we'll relate this infinite series to the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$, in two ways.

The extra factor $n+2$ suggests integration of x^{n+1} and xe^x:

$$\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\cdot n!}=\int_0^x u\sum_{n=0}^{\infty}\frac{u^n}{n!}\mathrm{d}u=\int_0^x ue^u\mathrm{d}u=xe^x-e^x+1;$$

the last equality is obtained by integrating by parts. This gives

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+2)\cdot n!\cdot 2^n}=-4\sum_{n=1}^{\infty}\frac{1}{(n+2)\cdot n!}\left(\frac{-1}{2}\right)^{n+2}=-4\left(\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\cdot n!}-\frac{x^{0+2}}{(0+2)\cdot 0!}\right)\bigg|_{x=-\frac{1}{2}}$$

$$=-4\left(xe^x-e^x+1-\frac{x^2}{2}\right)\bigg|_{x=-\frac{1}{2}}=\boxed{\frac{12e^{-1/2}-7}{2}}$$

Alternatively, we can complete the denominator to $(n+2)!$:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+2)\cdot n!\cdot 2^n}=-4\sum_{n=1}^{\infty}\frac{n+1}{(n+2)!}\left(\frac{-1}{2}\right)^{n+2}=-4\sum_{n=1}^{\infty}\frac{n+2-1}{(n+2)!}x^{n+2}\bigg|_{x=-\frac{1}{2}}$$

$$=-4\left(x\sum_{n=1}^{\infty}\frac{x^{n+1}}{(n+1)!}-\sum_{n=1}^{\infty}\frac{x^{n+2}}{(n+2)!}\right)\bigg|_{x=-\frac{1}{2}}=-4\left(x\sum_{n=2}^{\infty}\frac{x^n}{n!}-\sum_{n=3}^{\infty}\frac{x^n}{n!}\right)\bigg|_{x=-\frac{1}{2}}$$

$$=-4\left(x\left(e^x-\frac{x^0}{0!}-\frac{x^1}{1!}\right)-\left(e^x-\frac{x^0}{0!}-\frac{x^1}{1!}-\frac{x^2}{2!}\right)\right)\bigg|_{x=-\frac{1}{2}}$$

$$=-4\left(xe^x-e^x+1-\frac{x^2}{2}\right)\bigg|_{x=-\frac{1}{2}}=\boxed{\frac{12e^{-1/2}-7}{2}}$$

Grading: 2pts for adjustments in infinite series; integration of xe^x or completion to $(n+2)!$ 2pts; adjustment of lower limits 2pts; sum of power series 1pt; answer 1pt (any reasonable form ok)

Problem 7 (20pts)

Find the general real solution to each of the following differential equations.

(a; 6pts) $y'' = 0, \quad y = y(x)$

The associated polynomial equation is $r^2 = 0$. Thus, the two roots $r_1 = r_2 = 0$ are the same, and the general real solution is

$$y(x) = C_1 e^{0x} + C_2 x e^{0x} = \boxed{C_1 + C_2 x}$$

Grading: associated polynomial 2pts; roots 1pt; general solution 3pts; using integration twice is fine (with $y' = C$ 3pts).

(b; 7pts) $y'' + 4y' + 5y = 0, \quad y = y(x)$

The associated polynomial equation is

$$r^2 + 4r + 5 = 0 \qquad \Longleftrightarrow \qquad r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm \mathrm{i}.$$

Thus, the two roots are complex, and the general real solution is

$$y(x) = \boxed{C_1 e^{-2x} \cos(x) + C_2 e^{-2x} \sin(x) = e^{-2x}\big(C_1 \cos(x) + C_2 \sin(x)\big)}$$

Grading: associated polynomial 2pts; roots (in simplest form) 3pts; general solution 2pts (either form in the box is fine)

(c; 7pts) $y'' + 4y' - 5y = 0, \quad y = y(x)$

The associated polynomial equation is

$$r^2 + 4r - 5 = 0 \qquad \Longleftrightarrow \qquad (r+5)(r-1) = 0 \qquad \Longleftrightarrow \qquad r = 1, -5.$$

Since this polynomial has distinct real roots $r_1 = 1$ and $r_2 = -5$, the general solution is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} = \boxed{C_1 e^{x} + C_2 e^{-5x}}$$

Grading: associated polynomial 2pts; roots 3pts; general solution 2pts.

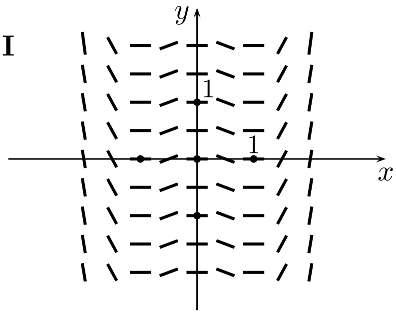
Problem 8 (10pts)

Consider the four differential equations for $y = y(x)$:

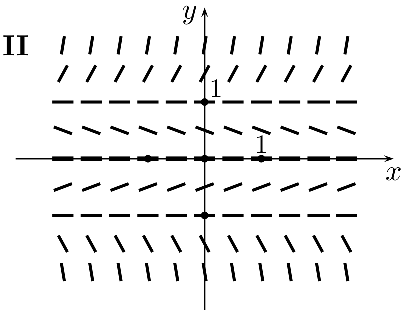
(a) $y' = x(x^2 - 1)$ (b) $y' = x(y^2 - 1)$ (c) $y' = y(x^2 - 1)$ (d) $y' = y(y^2 - 1)$.

Each of the two diagrams below shows the direction field for one of these equations:

I

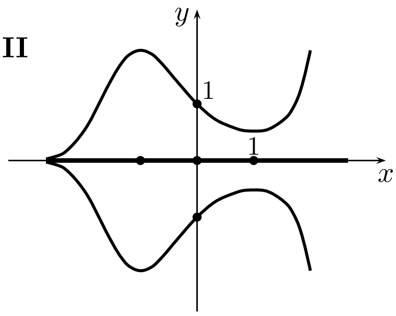


II

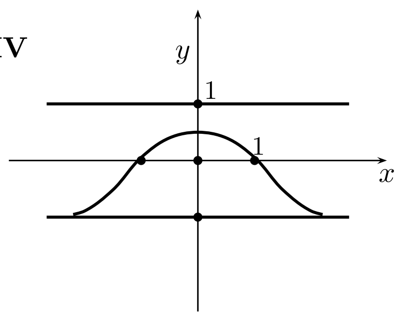


Each of the two diagrams below shows three solution curves for one of these equations:

III



IV



(so ALL three curves in diagram III are solution curves for either (a), or (b), or (c), or (d); same (?) for ALL three curves)

Match each of the diagrams to the corresponding differential equation (the match is one-to-one):

diagram	I	II	III	IV
equation	a	d	c	b

Answer Only: *no explanation is required.*

The slopes in I do not depend on y (do not change under vertical shifts); this is the case only for (a). The slopes in II do not depend on x (do not change under horizontal shifts); this is the case only for (d).

In III, $y = 0$ is a solution curve; this satisfies (c), but not (b); it also satisfies (d), but the upper solution curve in III has a nonzero slope at $(0, 1)$, which is contrary to (d). In IV, $y = 1$ and $y = -1$ are solution curves; these satisfy (b), but not (c); they also satisfy (d), but the middle solution curve has a nonzero slope at $(1, 0)$, which is contrary to (d).

Grading:

correct − repeats	0-	1	2	3	4
points	0	2	5	9	10

Problem 9 (10pts)

Answer Only. *A two-species interaction is modeled by the following system of differential equations*

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{1}{10}x^2 - \frac{1}{40}xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}y - \frac{1}{100}xy \end{cases} \qquad (x,y) = (x(t), y(t)),$$

where t *denotes time.*

(a; 2pts) *Which of the following best describes the interaction modeled by this system?*

(i) predator-prey (ii) competition for same resources (iii) cooperation for mutual benefit

Because of the coefficient of $-\frac{1}{40}$ in front of xy in the first equation, the x-species is hurt by the presence of the y-species (the growth rate of the former is reduced if the population of the latter is nonzero). Because of the coefficient of $-\frac{1}{100}$ in front of xy in the second equation, the y-species is hurt by the presence of the x-species.

Grading: correct answer circled 2pts; otherwise 0pts

(b; 8pts) *This system has 3 equilibrium (constant) solutions; find all of them and explain their significance relative to the interaction the system is modeling. Put one equilibrium solution in each box* below and *use the space to the right of the box to describe its significance.*

(0,0)	no population of either species ever
(10,0)	x-population at its carrying capacity in the absence of y-population
$(50, -160)$	meaningless, because the y-population is negative

The constant solutions are described by $(x'(t), y'(t)) = 0$. Using the above system this gives

$$\begin{cases} 0 = x\left(1 - \frac{1}{10}x - \frac{1}{40}y\right) \\ 0 = \frac{1}{2}y\left(1 - \frac{1}{50}x\right) \end{cases} \iff \begin{cases} x = 0 \ \text{ or } \ 1 - \frac{1}{10}x - \frac{1}{40}y = 0 \\ y = 0 \ \text{ or } \ 1 - \frac{1}{50}x = 0 \end{cases}$$

Thus, the constant solutions are the solutions of the following systems:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x = 0 \\ 1 - \frac{1}{50}x = 0 \end{cases} \qquad \begin{cases} 1 - \frac{1}{10}x - \frac{1}{40}y = 0 \\ y = 0 \end{cases} \qquad \begin{cases} 1 - \frac{1}{10}x - \frac{1}{40}y = 0 \\ 1 - \frac{1}{50}x = 0 \end{cases}$$

The first system gives the equilibrium in the first box. The second system has no solutions. The third system gives the equilibrium in the second box, by plugging in $y=0$ into the first equation. The last system gives the third equilibrium point by plugging in $x=50$ into the first equation.

Grading: 1 correct pair 1pt, 2 3pts, 3 5pts, with 1-2pts off for not simplifying; significance 1pt each

Problem 10A (20pts)

Only the higher of your scores on Problems 10A and 10B will count toward the total for the exam

A tank contains 150L of pure water. Brine (salt solution) containing .05 kg of salt per liter of water enters the tank at a rate of 10 L/min. In addition, brine containing .04 kg of salt per liter of water enters the tank at a rate of 5 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. Let $y(t)$ be the amount of salt in the tank, measured in kgs, after t minutes.

(a; 8pts) *Explain why the function $y=y(t)$ solves the initial-value problem*

$$y' = \frac{7}{10} - \frac{1}{10}y, \quad y = y(t), \qquad y(0) = 0.$$

Since there is initially no salt in the tank, $y(0)=0$. Furthermore, $y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t)$, where

$$\begin{aligned} y'_{\text{in}}(t) &= (\text{flow rate of salt})_{\text{in}} = (\text{flow rate of solution})_{\text{in},1} \cdot (\text{salt concentration})_{\text{in},1} \\ &\qquad + (\text{flow rate of solution})_{\text{in},2} \cdot (\text{salt concentration})_{\text{in},2} = 10 \cdot .05 + 5 \cdot .04 = .7; \\ y'_{\text{out}}(t) &= (\text{flow rate of salt})_{\text{out}} = (\text{flow rate of solution})_{\text{out}} \cdot (\text{salt concentration})_{\text{out}}. \end{aligned}$$

Since the salt in the tank is thoroughly mixed, the outgoing salt concentration is the same as the salt concentration in the tank:

$$(\text{salt concentration})_{\text{out}} = \frac{\text{amount salt in tank}}{\text{volume in tank}} = \frac{y(t)}{150},$$

since the volume of solution in the tank is kept constant at 150 gallons. So,

$$y'_{\text{out}}(t) = 15 \cdot \frac{y(t)}{150} = \frac{y(t)}{10}.$$

It follows that $y(t)$ is a solution to the differential equation $y' = .7 - y/10$.

Grading: mention of initial condition 1pt; computation of y'_{in} 3pts and of y'_{out} 2pts; remainder 2pts; mostly formulas is ok

(b; 8pts) *Find the solution $y=y(t)$ to the initial-value problem stated in (a).*

First find the general solution to the DE. Since it is separable, writing $y' = \mathrm{d}y/\mathrm{d}t$, moving everything involving y to LHS and everything involving t to the RHS, and integrating, we obtain

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{7-y}{10} \quad &\iff \quad \frac{\mathrm{d}y}{7-y} = \frac{\mathrm{d}t}{10} \quad \iff \quad \int \frac{\mathrm{d}y}{7-y} = \int \frac{\mathrm{d}t}{10} \quad \iff \quad -\ln|7-y| = \frac{t}{10} + C \\ &\iff \quad \ln|7-y| = -\frac{t}{10} - C \quad \iff \quad \mathrm{e}^{\ln|7-y|} = \mathrm{e}^{-t/10-C} = \mathrm{e}^{-C}\mathrm{e}^{-t/10} \\ &\iff \quad |7-y| = A\mathrm{e}^{-t/10} \quad \iff \quad 7-y = \pm A\mathrm{e}^{-t/10} \quad \iff \quad y(t) = 7 - C\mathrm{e}^{-t/10}. \end{aligned}$$

Plugging in the initial condition $(t,y) = (0,0)$, we obtain $0 = 7 - C\mathrm{e}^{-0/10} = 7 - C$, or $C = 7$. So $\boxed{y(t) = 7 - 7\mathrm{e}^{-t/10} = 7(1-\mathrm{e}^{-t/10})}$

Grading: separating variables 2pts; integration 1pt; remainder of computation to general solution 2pts; determining C 2pts; final answer 1pt

(c; 4pts) *How long will it take for the amount of salt in the tank to reach 3.5 kgs?*

We need to find t so that $y(t) = 3.5$, i.e. $3.5 = 7(1-\mathrm{e}^{-t/10})$. This gives

$$\frac{1}{2} = 1 - \mathrm{e}^{-t/10} \quad \iff \quad \mathrm{e}^{-t/10} = \frac{1}{2} \quad \iff \quad -\frac{t}{10} = \ln(1/2) = -\ln 2 \quad \iff \quad \boxed{t = 10(\ln 2) \ \text{ mins}}$$

Grading: setup 1pt; numerical solution 2pts; units 1pt

Problem 10B (20pts)

Only the higher of your scores on Problems 10A and 10B will count toward the total for the exam

(a; 8pts) *Show that the orthogonal trajectories to the family of curves $y^3 = kx^2$ are described by the differential equation*

$$y' = -\frac{3}{2}\frac{x}{y}, \qquad y = y(x).$$

Differentiate $y^3 = kx^2$ with respect to x, using chain rule and remembering that k is a constant:

$$3y^2y' = 2kx \qquad \Longleftrightarrow \qquad y' = \frac{2kx}{3y^2}.$$

From the original equation, we find that $k = y^3/x^2$ and so our curves have slope

$$y' = k\frac{2x}{3y^2} = \frac{y^3}{x^2} \cdot \frac{2x}{3y^2} = \frac{2y}{3x}$$

at (x, y). The slopes of the orthogonal curves are the negative reciprocal of this; so they satisfy

$$y' = -\frac{1}{2y/3x} = -\frac{3x}{2y}.$$

Grading: computation of slopes of the initial curves 4pts; the negative reciprocal statement 3pts; conclusion 1pt

(b; 6pts) *Find the general solution to the differential equation stated in (a).*

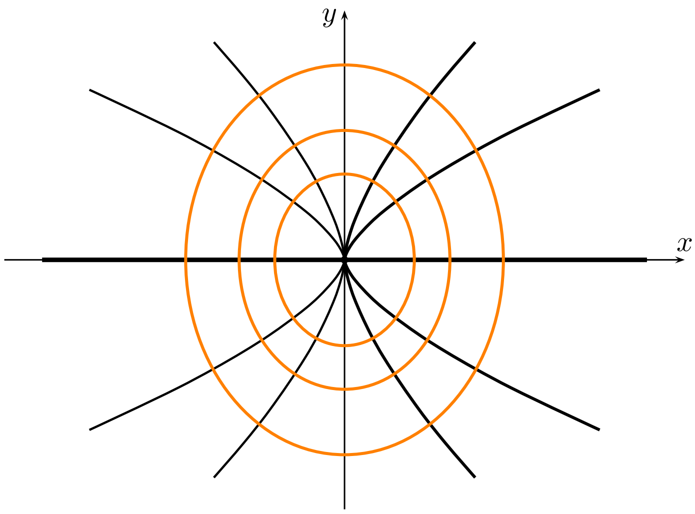
This equation is separable, so after writing $y' = \mathrm{d}y/\mathrm{d}x$, we can move everything involving y to LHS and everything involving x to RHS and then integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x}{2y} \iff 2y\,\mathrm{d}y = -3x\,\mathrm{d}x \iff \int 2y\,\mathrm{d}y = -\int 3x\,\mathrm{d}x \iff y^2 = -\frac{3}{2}x^2 + C$$
$$\iff \boxed{3x^2 + 2y^2 = C}$$

Grading: splitting the variables 2pts; integration 2pts; simplification 2pts (answer with both square roots is ok) (c; 6pts) *Sketch at least 3 representatives of the original family of curves and at least 3 orthogonal trajectories on the same diagram; indicate clearly which is which.*

Draw the above curves for different values of k and C:

- $y^3 = 0x^2$ is the x-axis; $y^3 = 1x^2$ is the graph of the function $y = x^{2/3}$, which has a sharp point at (0,0); $y^3 = (-1)x^2$ is the graph of the function $y = -x^{2/3}$ and is the reflection of the above graph about the x-axis;
- the equation $3x^2 + 2y^2 = C$ has no solutions (x, y) if $C < 0$; if $C = 0$, this "curve" is just the origin $(0, 0)$; the curve $3x^2 + 2y^2 = 6$ is the ellipse passing through the points $(\pm\sqrt{2}, 0)$ and $(0, \pm\sqrt{3})$, and so it is stretched vertically; the ellipse $3x^2+2y^2 = 6r^2$ is the same ellipse scaled by the factor $|r|$.



Grading: 1pt for each relevant curve; 2pts off if no indication is given which curves are which (k, C values not required); 1pt off if the axes are not labeled