MAT 127 Final Exam

December 13, 2010 8:15-10:45am

Name:______________________________ **ID:**____________________

first name last name

Section:	L01	L02	L03	(circle yours)
	MWF 9:35-10:30am	TuTh 5:20-6:40pm	TuTh 2:20-3:40pm	

DO NOT OPEN THIS EXAM YET

Instructions

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no calculators, no phones.

(3) Please write legibly to receive credit. Circle or box your final answers. If your solution to a problem does not fit on the page on which the problem is stated, please indicate on that page where in the exam to find (the rest of) your solution.

(4) You may continue your solutions on additional sheets of paper provided by the proctors. If you do so, please write your name and ID number at the top of each of them and staple them to the back of the exam (stapler available); otherwise, these sheets may get lost.

(5) Anything handed in will be graded; incorrect statements will be penalized even if they are in addition to complete and correct solutions. If you do not want something graded, please erase it or cross it out.

(6) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4) and simplify them as much as possible (e.g. $1/2$, not $2/4$) to receive full credit.

(7) Show your work; correct answers only will receive only partial credit (unless noted otherwise).

(8) Be careful to avoid making grievous errors that are subject to heavy penalties.

(9) If you need more blank paper, ask a proctor.

Out of fairness to others, please **stop working and close the exam as soon as the time is called**. A significant number of points will be taken off your exam score if you continue working after the time is called. You will be given a two-minute warning before the end.

Some Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if} \quad |x| < 1 \qquad\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad\qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

to receive full credit, justify any other power series expansion you use

1 (10pts)	
2 (10pts)	
3ab+c (20pts)	
4 (20pts)	
5 (20pts)	
Subtotal (80pts)	

6abc+d (10pts)	
7 (20pts)	
8 (10pts)	
9 (10pts)	
10A/B (20pts)	
Subtotal (70pts)	

Total (150pts)	

Problem 1 (10pts)

Answer Only. Determine whether each of the following sequences or series converges or not. In each case, *clearly* circle either **YES** or **NO**, but not both. Each correct answer is worth 2 points. You may use the blank space between the questions to figure out the answer, but no partial credit will be awarded and no justification is expected for your answers on this problem.

(a) the sequence $a_n = 1 + \dfrac{\cos^3 n}{n}$ **YES** **NO**

(b) the sequence $a_n = n^2(1 - e^{1/n})$ **YES** **NO**

(c) the series $\displaystyle\sum_{n=1}^{\infty} \frac{n + (-1)^n}{n^2 + 1}$ **YES** **NO**

(d) the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n + 1}$ **YES** **NO**

(e) the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{3^n + 5^n}}$ **YES** **NO**

Problem 2 (10pts)

Answer Only. Put your answer to each question in the corresponding box *in the simplest possible form.* No credit will be awarded if the answer in the box is wrong; partial credit may be awarded if the answer in the box is correct, but not in the simplest possible form.

(a; 5pts) Write the number $1.1\overline{09} = 1.1090909\ldots$ as a simple fraction

(b; 5pts) Find the limit of the sequence recursively defined by

$$a_1 = 4, \qquad a_{n+1} = 4 - \frac{3}{a_n} \quad n \geq 1.$$

Assume that this sequence converges.

Problem 3 (20pts)

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence.

(a; 10pts) $f(x) = x^2 + 2x$ around $x = -2$

(b; 10pts) $f(x) = \dfrac{x}{4 + x^2}$ around $x = 0$

(c; bonus 10pts)[1] $f(x) = \dfrac{1}{5 - 12x^2 + 4x^4}$ around $x = 0$

[1] this part is relatively hard and subject to harsh grading; do and double-check the rest of the exam first

Problem 4 (20pts)

(a; 8pts) Find the radius and interval of convergence of the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

(b; 4pts) Find $\lim_{x \longrightarrow 0} \frac{f(x) - x}{x^2}$

(c; 8pts) Find the Taylor series expansion for the function $g = g(x)$ given by

$$g(x) = \int_0^x \frac{f(u) - u}{u^2} \mathrm{d}u$$

around $x = 0$. What are the radius and interval of convergence of this power series?

Problem 5 (20pts)

Show that the following series are convergent and find their sums.

(a; 10pts) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{9^n(2n)!}$

(b; 10pts) $\displaystyle\sum_{n=1}^{\infty}\frac{n2^n}{3^n}$

Problem 6 (10pts)

All questions in this problem refer to the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+2) \cdot n! \cdot 2^n}$$

(a; 3pts) Explain why this series converges.

(b; 4pts) What is the minimal number of terms required to approximate the sum of this series with error less than 1/1000? Justify your answer.

(c; 3pts) Based on your answer in part (b), estimate the sum of this series with error less than 1/1000; leave your answer as a simple fraction p/q for some integers p and q with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why. (If you do not know how to do (b), take the answer to (b) to be 2).

(d; bonus 8pts)[2] Find the sum of the infinite series exactly.

[2]this part is relatively hard and subject to harsh grading; do and double-check the rest of the exam first

Problem 7 (20pts)

Find the general real solution to each of the following differential equations.

(a; 6pts) $y'' = 0, \quad y = y(x)$

(b; 7pts) $y'' + 4y' + 5y = 0, \quad y = y(x)$

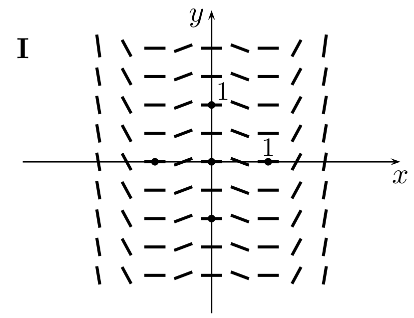
(c; 7pts) $y'' + 4y' - 5y = 0, \quad y = y(x)$

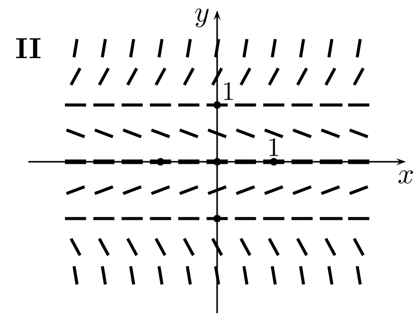
Problem 8 (10pts)

Consider the four differential equations for $y = y(x)$:

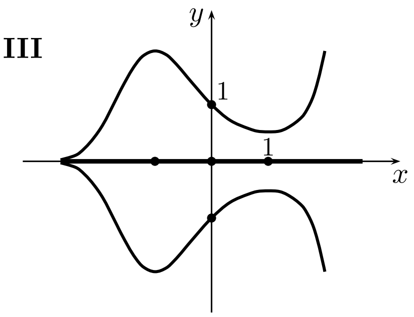
(a) $y' = x(x^2 - 1)$ (b) $y' = x(y^2 - 1)$ (c) $y' = y(x^2 - 1)$ (d) $y' = y(y^2 - 1)$.

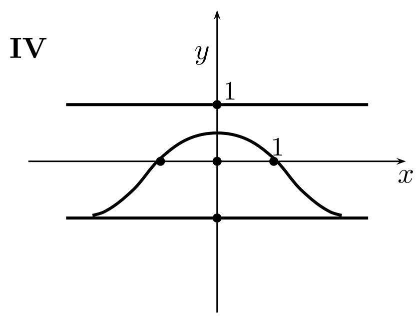
Each of the two diagrams below shows the direction field for one of these equations:





Each of the two diagrams below shows three solution curves for one of these equations:





(so ALL three curves in diagram III are solution curves for either (a), or (b), or (c), or (d); same (?) for ALL three curves)

Match each of the diagrams to the corresponding differential equation (the match is one-to-one):

diagram	I	II	III	IV
equation				

Answer Only: no explanation is required.

do not write below this line or your work on this problem will be void

grader's use only

correct − repeats	0-	1	2	3	4
points	0	2	5	9	10

Problem 9 (10pts)

Answer Only. A two-species interaction is modeled by the following system of differential equations

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{1}{10}x^2 - \frac{1}{40}xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}y - \frac{1}{100}xy \end{cases} \qquad (x, y) = (x(t), y(t)),$$

where t denotes time.

(a; 2pts) Which of the following best describes the interaction modeled by this system?

(i) predator-prey (ii) competition for same resources (iii) cooperation for mutual benefit

Circle your answer above.

(b; 8pts) This system has 3 equilibrium (constant) solutions; find all of them and explain their significance relative to the interaction the system is modeling. *Put one equilibrium solution in each box* below and *use the space to the right of the box to describe its significance.*

[ ] ______________________________

[ ] ______________________________

[ ] ______________________________

You can use the space below (or elsewhere) to figure out your answers, but your score will based on your **answers above only**.

Problem 10A (20pts)

Only the higher of your scores on Problems 10A and 10B will count toward the total for the exam

A tank contains 150L of pure water. Brine (salt solution) containing .05 kg of salt per liter of water enters the tank at a rate of 10 L/min. In addition, brine containing .04 kg of salt per liter of water enters the tank at a rate of 5 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. Let $y(t)$ be the amount of salt in the tank, measured in kgs, after t minutes.

(a; 8pts) Explain (based on the above information) why the function $y=y(t)$ solves the initial-value problem

$$y' = \frac{7}{10} - \frac{1}{10}y, \quad y = y(t), \qquad y(0) = 0.$$

(b; 8pts) Find the solution $y=y(t)$ to the initial-value problem stated in (a).

(c; 4pts) How long will it take for the amount of salt in the tank to reach 3.5 kgs?

Problem 10B (20pts)

Only the higher of your scores on Problems 10A and 10B will count toward the total for the exam

(a; 8pts) Show that the orthogonal trajectories to the family of curves $y^3 = kx^2$ are described by the differential equation

$$y' = -\frac{3}{2}\frac{x}{y}\,, \qquad y = y(x).$$

(b; 6pts) Find the general solution to the differential equation stated in (a).

(c; 6pts) Sketch at least 3 representatives of the original family of curves and at least 3 orthogonal trajectories on the same diagram; indicate clearly which is which.