MAT 127 Final Exam

May 11, 2022 8:15-10:45am

Name:______________________________ **ID:**____________________

first name *first*

Section:	L01	L02	L03	(circle yours)
	MW 4:25-5:45pm	TuTh 9:45-11:05am	TuTh 1:15-2:35pm	

DO NOT OPEN THIS EXAM YET

Instructions

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no calculators, no phones.

(3) Please write legibly to receive credit. Circle or box your final answers. If your solution to a problem does not fit on the page on which the problem is stated, please indicate on that page where in the exam to find (the rest of) your solution.

(4) You may continue your solutions on additional sheets of paper provided by the proctors. If you do so, please write your name and ID number at the top of each of them and staple them to the back of the exam (stapler available); otherwise, these sheets may get lost.

(5) Anything handed in will be graded; incorrect statements will be penalized even if they are in addition to complete and correct solutions. If you do not want something graded, please erase it or cross it out.

(6) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4) and simplify them as much as possible (e.g. $1/2$, not $2/4$) to receive full credit.

(7) Show your work; correct answers only will receive only partial credit (unless noted otherwise).

(8) Be careful to avoid making grievous errors that are subject to heavy penalties.

(9) If you need more blank paper, ask a proctor.

Out of fairness to others, please **stop working and close the exam as soon as the time is called**. A significant number of points will be taken off your exam score if you continue working after the time is called. You will be given a two-minute warning before the end.

Some Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if} \ \ |x| < 1 \qquad\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad\qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

to receive full credit, justify any other power series expansion you use

1 (10pts)	
2 (10pts)	
3 (10pts)	
4 (15pts)	
5ab+c (20pts)	
6 (10pts)	
Subtotal (75pts)	

7abc+d (10pts)	
8 (10pts)	
9+bn (10pts)	
10 (15pts)	
11ab+c (10pts)	
12A/B (20pts)	
Subtotal (75pts)	

Total (150pts)	

Problem 1 (10pts)

Answer Only. Determine whether each of the following sequences or series converges or not. In each case, *clearly* circle either **YES** or **NO**, but not both. Each correct answer is worth 2 points. You may use the blank space between the questions to figure out the answer, but no partial credit will be awarded and no justification is expected for your answers on this problem.

(a) the sequence $a_n = (-1)^n \sin(1/n)$ **YES** **NO**

(b) the sequence $a_n = n^2\big(1 - \cos(1/n)\big)$ **YES** **NO**

(c) the series $\sum_{n=1}^{\infty}(-1)^n$ **YES** **NO**

(d) the series $\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n)}{n^2+1}$ **YES** **NO**

(e) the series $\sum_{n=1}^{\infty} \frac{2^n+7^n}{5^n+6^n}$ **YES** **NO**

Problem 2 (10pts)

Answer Only. Suppose $c_0, c_1, c_2, \ldots \geq 0$, the series $\sum_{n=0}^{\infty} c_n 3^n$ converges, and $\sum_{n=0}^{\infty} c_n(-5)^n$ diverges. What can be said about the convergence of the infinite series below? In each case, *clearly* circle exactly *one* of the three choices. Each correct answer is worth 1 point.

$\sum_{n=0}^{\infty} c_n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n 7^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} c_n 7^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} n c_n 2^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} c_n 4^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n 4^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n 3^n$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} c_n^2$:	**converges**	**diverges**	**impossible to tell**
$\sum_{n=0}^{\infty} (-1)^n c_n^2 4^n$:	**converges**	**diverges**	**impossible to tell**

Problem 3 (10pts)

Answer Only. Put your answer to each question in the corresponding box *in the simplest possible form.* No credit will be awarded if the answer in the box is wrong; partial credit may be awarded if the answer in the box is correct, but not in the simplest possible form.

(a; 5pts) Write the number $1.\overline{06} = 1.060606\ldots$ as a simple fraction

(b; 5pts) Find the limit of the sequence

$$\sqrt{12},\ \sqrt{12+\sqrt{12}},\ \sqrt{12+\sqrt{12+\sqrt{12}}},\ \sqrt{12+\sqrt{12+\sqrt{12+\sqrt{12}}}},\ \ldots$$

Assume that this sequence converges.

Problem 4 (15pts)

All questions in this problem refer to the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{2^n x^n}{n}.$$

(a; 9pts) Find the radius and interval of convergence of this power series.

(b; 6pts) Find the Taylor series expansion for the function $f'(x)$ around $x=0$. What are the radius and interval of convergence of this power series?

Problem 5 (20pts)

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence.

(a; 10pts) $f(x) = x^2$ around $x = 2$

(b; 10pts) $f(x) = \dfrac{2x^2}{8 - x^3}$ around $x = 0$

(c; bonus 10pts) $f(x) = \dfrac{1}{1 - x^2 - 2x^4}$ around $x = 0$

Warning. The bonus part is hard and is subject to harsh grading. Your time is likely to be better spent double- and triple-checking your work on the rest of the exam. If you do not see better ways of using your time on the exam, please answer this question either on the facing page or on the back of this page and state below where to find the answer.

Problem 6 (10pts)

Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (\ln 5)^n}{n!}$$

converges and find its sum.

Problem 7 (10pts)

All questions in this problem refer to the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1) \cdot (2n)! \cdot 2^n}$$

(a; 3pts) Explain why this series converges.

(b; 4pts) What is the minimal number of terms required to approximate the sum of this series with error less than 1/1000? Justify your answer.

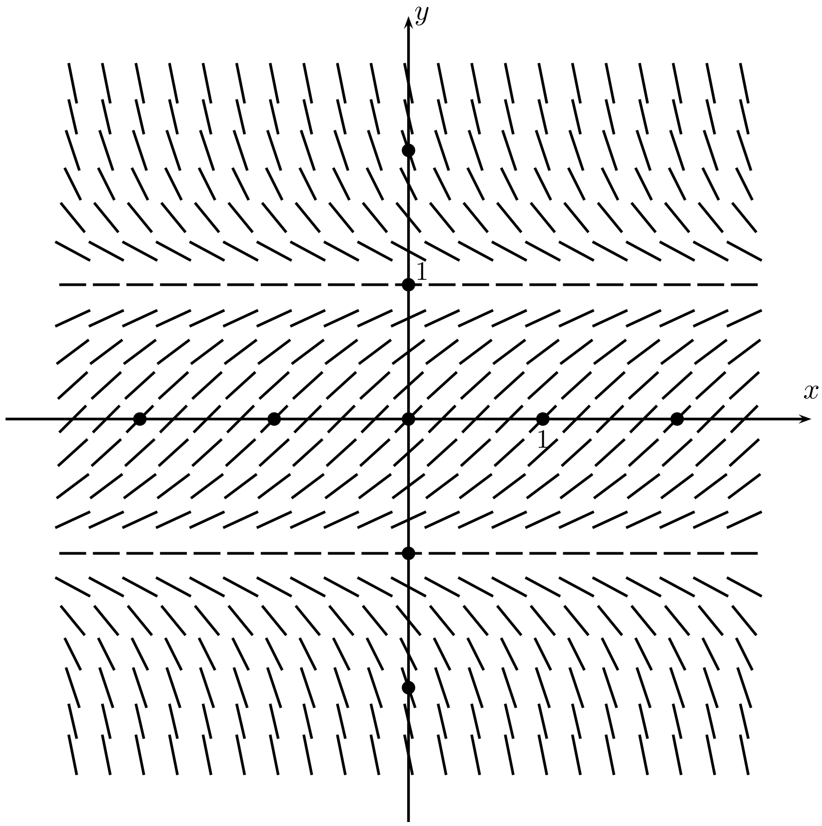
(c; 3pts) Based on your answer in part (b), estimate the sum of this series with error less than 1/1000; leave your answer as a simple fraction p/q for some integers p and q with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why. (If you do not know how to do (b), state a guess for the answer in (b)).

(d; bonus 10pts) Find the sum of the infinite series exactly.

Warning. The bonus part is hard and is subject to harsh grading. Your time is likely to be better spent double- and triple-checking your work on the rest of the exam. If you do not see better ways of using your time on the exam, please answer this question either on the facing page or on the back of this page and state below where to find the answer.

Problem 8 (10pts)

The direction field for a differential equation is shown below.



(a; 8pts) On the direction field, sketch and *clearly label* the graphs of the four solutions with the initial conditions $y(0) = 0$, $y(0) = .5$, $y(0) = -1$, and $y(0) = -2$ (each of these four conditions determines a solution to the differential equation). *No explanation is required.*

(b; 2pts) The direction field above is for one of the following differential equations for $y = y(x)$:

$$\text{(i) } y' = 1 - x^4\,, \qquad \text{(ii) } y' = 1 - y^4\,, \qquad \text{(iii) } y' = 1 - x^4 - y^4\,.$$

Which of these three equations does the direction field correspond? *Circle your answer above. No explanation is required.*

Problem 9 (10pts)

Let $f=f(x)$ and $g=g(x)$ be two solutions of the differential equation

$$y'' + 2y' + y = \mathrm{e}^{-x}, \qquad y = y(x).$$

Show that $h=f-g$ is a solution of the differential equation

$$y'' + 2y' + y = 0, \qquad y = y(x).$$

Show your work and/or explain your reasoning.

(bonus 15pts) Find the general solution of the first differential equation above and use it to show that $h=f-g$ is a solution of the second differential equation.

Warning. The bonus part is hard and is subject to harsh grading. Your time is likely to be better spent double- and triple-checking your work on the rest of the exam. If you do not see better ways of using your time on the exam, please answer this question either on the facing page or on the back of this page and state below where to find the answer.

Problem 10 (15pts)

Answer Only. Find the general real solution to each of the following differential equations. Put your answer to each question in the corresponding box *in the simplest possible form.* Each correct answer in the simplest possible form is worth 5pts. Partial credit may be awarded for not completely correct answers.

(a) $y'' + 2y' + y = 0, \quad y = y(x)$

(b) $y'' + 2y' = 0, \quad y = y(x)$

(c) $y'' + y = 0, \quad y = y(x)$

Problem 11 (10pts)

Answer Only. A two-species interaction is modeled by the following system of differential equations

$$\begin{cases} \frac{dx}{dt} = x - \frac{1}{10}x^2 + \frac{1}{100}xy \\ \frac{dy}{dt} = \frac{1}{20}y - \frac{1}{100}y^2 + \frac{1}{20}xy \end{cases} \qquad (x,y) = (x(t), y(t)),$$

where t denotes time.

(a; 2pts) Which of the following best describes the interaction modeled by this system?

(i) predator-prey (ii) competition for same resources (iii) cooperation for mutual benefit

Circle your answer above.

(b; 8pts) This system has 4 equilibrium (constant) solutions; find all of them. *Put one equilibrium solution in each box below (no explanation of significance is required).*

(c; bonus 10pts) Let $(x, y) = (x(t), y(t))$ be a solution of the system above such that $x(0), y(0) > 0$. Find $\lim_{t \longrightarrow \infty} x(t)$.

Warning. *The bonus part is hard and is subject to harsh grading. Your time is likely to be better spent double- and triple-checking your work on the rest of the exam. If you do not see better ways of using your time on the exam, please answer this question either on the facing page or on the back of this page and state below where to find the answer.*

Problem 12A (20pts)

Only the higher of your scores on Problems 12A and 12B will count toward the total for the exam

A tank contains 100 liters of salt solution with 400 grams of salt dissolved in it. A salt solution containing 6g of salt per liter enters the tank at a rate of 5 liters per minute. The solution is kept thoroughly mixed and drains at a rate of 5L/min (so the volume in the tank stays constant). Let $y(t)$ be the amount of salt in the tank, measured in grams, after t minutes.

(a; 8pts) Explain (based on the above information) why the function $y = y(t)$ solves the initial-value problem

$$y' = 30 - \frac{y}{20}, \quad y = y(t), \qquad y(0) = 400.$$

(b; 8pts) Find the solution $y = y(t)$ to the initial-value problem stated in (a).

(c; 4pts) How long will it take for the amount of salt in the tank to reach 500 grams?

Problem 12B (20pts)

Only the higher of your scores on Problems 12A and 12B will count toward the total for the exam

A lake with carrying capacity of 1,000 fish is stocked with 100 fish. The number of fish grows according to the logistic equation and doubles in 4 months.

(a; 12pts) Show that the number of fish in the lake after t months is given by

$$P(t) = \frac{1,000}{1 + 9(2/3)^{t/2}} \, .$$

(b; 8pts) How long will it take for the fish population to reach 500? Express your answers in terms of natural logs and simplify as much as possible.