Problems before Mid II

1. Is the sequence convergent or divergent? If convergent, what is the limit?

a) $\dfrac{5^n + n! + n^3}{n^2 + 2^n + n^n}$ b) $\dfrac{3n+5}{2+\sqrt{n^2+1}}$ c) $\dfrac{(-1)^n n}{2n+\sqrt{n^2-1}}$

d) $\dfrac{\arctan(n^2)}{\sqrt{n}}$ e) $\dfrac{n}{n+1} + \dfrac{\cos(n)}{n}$ f) $\sqrt{9^n + 2^n} - 3^n$

2. Is the series convergent or divergent? (We do not want to find the value.)

a) $\sum_{n=1}^{\infty} \dfrac{2n}{\sqrt{n^2+1}}$ b) $\sum_{n=1}^{\infty} \dfrac{n^2}{3^n}$ c) $\sum_{n=1}^{\infty} \dfrac{n!}{n^n}$

d) $\sum_{n=1}^{\infty} \dfrac{n^3+5^n}{3^n+\sqrt{n}}$ e) $\sum_{n=1}^{\infty} \dfrac{\ln(n)+n}{\sqrt{n^3+1}+n}$ f) $\sum_{n=1}^{\infty} \tan\left(\dfrac{1}{n^2}\right)$

g) $\sum_{n=1}^{\infty} \ln\left(1+\dfrac{1}{2^n}\right)$ h) $\sum_{n=1}^{\infty} \dfrac{|\sin(n)|}{n^2}$ i) $\sum_{n=1}^{\infty} (\cos 1)^n$

j) $\sum_{n=1}^{\infty} (\cos(n) - \cos(n+1))$ k) $\sum_{n=1}^{\infty} \left(\cos\left(\dfrac{1}{n}\right) - \cos\left(\dfrac{1}{n+1}\right)\right)$ l) $\sum_{n=1}^{\infty} \left(\sqrt{9^n+2^n} - 3^n\right)$

3. Is the series convergent or divergent? If convergent, find its value.

a) $\sum_{n=4}^{\infty} \dfrac{1}{n^2-4n+3}$ b) $\sum_{n=2}^{\infty} \left(\dfrac{1}{e^n} + \dfrac{1}{n(n-1)}\right)$

c) $\sum_{n=1}^{\infty} \dfrac{3^{n-1}}{2^{2n}}$ d) $\sum_{n=1}^{\infty} \dfrac{2^{n-1}+3^{n+1}}{5^n}$

4. For which values of x is the following series convergent? For a given such x, what is the value of the series (as a function of x)?

a) $\sum_{n=1}^{\infty} \dfrac{(x+3)^n}{2^{2n}}$ b) $\sum_{n=1}^{\infty} 2^n \cos^n x$ c) $\sum_{n=1}^{\infty} (1+x)^{-n}$

5. Is the **series** $\sum_{n=1}^{\infty} a_n$ convergent or divergent if the **sequence** a_n is defined recursively as:

a) $a_1 = 1, a_{n+1} = \dfrac{5n+1}{4n+3} a_n$ b) $a_1 = 1, a_{n+1} = \dfrac{2+\cos(n)}{\sqrt{n}} a_n$

6. Is the series convergent or divergent? Is it absolutely convergent? (In general, for the majority of given series here, you will need *both* the alternating series test and the comparison / limit comparison test to answer *both* questions. Working on the problems in the given order might be helpful.)

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$ g) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right)}{n}$ h) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2}$

i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\cos(n\pi)}{n}$ j) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\cos(n\pi)}{n^2}$ k) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\cos\left(\frac{n\pi}{2}\right)}{n}$ l) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\cos\left(\frac{n\pi}{2}\right)}{n^2}$

m) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ n) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n^2}\right)$ o) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin^2\left(\frac{1}{n}\right)$ p) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$

q) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{(-1)^n}{n}\right)$ r) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{(-1)^n}{n^2}\right)$ s) $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{(-1)^n}{n}\right)$ t) $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{(-1)^n}{n}\right)$

7. Is the series convergent or divergent? Is it absolutely convergent?

a)* $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(1-\cos\left(\frac{1}{n}\right)\right)$ b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n}{n+1}\right)$ c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^4-n^2+1}{n^5-5n+1}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\arctan(n^2)}{\sqrt{n}}$ e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}+(-1)^n}{\sqrt{n}}$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+\sin(n)}{n^2}$

**Hint for* 2a):

$$(1-\cos(x))(1+\cos(x)) = 1-\cos^2(x) = \sin^2(x)$$

$$\lim_{x\to 0} \frac{\sin^2(x)}{x^2} = 1$$

Answer Key

1. (a) converges to 0 (divide top and bottom by n^n, which dominates)
   (b) converges to 3 (divide top and bottom by n)
   (c) diverges (divide top and bottom by n; odd terms approach $-1/3$, even $1/3$)
   (d) converges to 0 (arctan takes values in $(-\pi/2, \pi/2)$)
   (e) converges to 1 (divide the top and bottom in the first fraction by n)
   (d) converges to 0 (multiply and divide by $\sqrt{9^n+2^n}+3^n$; divide top and bottom by 3^n)

2. (a) $\sum$ diverges by Test for Divergence (the terms a_n approach 2)
   (b) converges by Ratio Test ($|a_{n+1}|/|a_n|$ approaches $1/3$)
   (c) converges by RT ($|a_{n+1}|/|a_n|$ approaches $1/e$)
   (d) diverges by RT ($|a_{n+1}|/|a_n|$ approaches $5/3$)
   (e) diverges by Limit Comparison Test with $b_n = n/\sqrt{n^3} = 1/n^{1/2}$ and p-Series Test
   (f) converges by LCT with $b_n = 1/n^2$ and p-Series Test
   (g) converges by LCT with $b_n = 1/2^n$ and the geometric series test or RT ($|a_{n+1}|/|a_n|$ approaches $1/2$)
   (h) converges by Comparison Test with $b_n = 1/n^2$ and p-Series Test
   (i) converges by the geometric series test
   (j) diverges b/c $s_n = \cos(1) - \cos(n+1)$ diverges
   (k) converges b/c $s_n = \cos(1/1) - \cos(1/(n+1))$ converges
   (l) converges by RT ($|a_{n+1}|/|a_n|$ approaches $2/3$) or LCT with $b_n = (2/3)^n$

3. (a) converges to $3/4$ b/c partial fractions and pairwise cancellation give

$$s_n = \frac{1}{2}\left(\frac{1}{4-3} + \frac{1}{5-3} - \frac{1}{(n-1)-1} - \frac{1}{n-1}\right)$$

   (b) converges to $1/e(e-1)+1$ b/c the first part is a geometric series with $r = 1/e$ and $a_0 = 1/e^2$, while partial fractions and pairwise cancellation give $s_n = \dfrac{1}{2-1} - \dfrac{1}{n}$ for the second part
   (c) converges to 1 b/c it is a geometric series with $r = 3/4$ and $a_0 = 1/4$
   (d) converges to $1/3+9/2 = 29/6$ b/c the first part is a geometric series with $r = 2/5$ and $a_0 = 1/5$, while the second part is a geometric series with $r = 3/5$ and $a_0 = 9/5$

4. (a) this is a geometric series with $r = (x+3)/4$ and $a_0 = (x+3)/4$; it converges to $(x+3)/(1-x)$ if $x \in (-7, 1)$; it diverges otherwise
   (b) this is a geometric series with $r = 2\cos x$ and $a_0 = 2\cos x$; it converges to $2\cos x/(1-2\cos x)$ if $\frac{\pi}{3} + \pi k < x < \frac{2\pi}{3} + \pi k$ for an integer k
   (c) this is a geometric series with $r = 1/(1+x)$ and $a_0 = 1/(1+x)$; it converges to $1/x$ if $x < -2$ or $x > 0$

5. (a) diverges by Ratio Test ($|a_{n+1}|/|a_n|$ approaches $5/4$)
   (b) converges by Ratio Test ($|a_{n+1}|/|a_n|$ approaches 0)

6. (a) converges by Alternating Series Test, but not absolutely
(b) converges by AST or Absolute Convergence Test (+ p-Series)
(c) converges by AST, but not absolutely
(d) converges by AST or ACT (+ Limit Comparison with $b_n = 1/n^2$ and p-series)
(e) converges by AST, but not absolutely
(f) converges by AST or ACT (+ Comparison with $b_n = 1/n^2$ and p-series)
(g) converges by AST after dropping $a_{2k+1} = 0$, but not absolutely
(h) converges by AST after dropping $a_{2k+1} = 0$ or ACT (+ Comparison with $b_n = 1/n^2$)
(i) diverges by p-series (b/c $(-1)^n \cos(\pi n) = 1$)
(j) converges by ACT (+ Comparison with $b_n = 1/n^2$ and p-series)
(k) converges by AST after dropping $a_{2k+1} = 0$ (b/c $(-1)^{2k} \cos(\pi(2k)/2) = (-1)^k$), but not absolutely
(l) converges by AST or ACT (+ Comparison with $b_n = 1/n^2$ and p-series)
(m) converges by AST, but not absolutely (LCT with $b_n = 1/n$)
(n) converges by AST or ACT (+ LCT with $b_n = 1/n^2$ and p-series)
(o) converges by AST or ACT (+ LCT with $b_n = 1/n^2$ and p-series)
(p) diverges by Test for Divergence (odd terms approach -1, even $+1$)
(q) converges by AST, but not absolutely (LCT with $b_n = 1/n$)
(r) converges by AST or ACT (+ LCT with $b_n = 1/n^2$ and p-series)
(s) converges by ACT (+ LCT with $b_n = 1/n^2$ and p-series)
(t) converges by AST, but not absolutely (LCT with $b_n = 1/n$)

7. (a) converges by AST or ACT (+ LCT with $b_n = 1/n^2$ and p-series)
(b) converges by AST, but not absolutely (LCT with $b_n = 1/n$)
(c) converges by AST, but not absolutely (LCT with $b_n = 1/n$)
(d) converges by AST, but not absolutely (LCT with $b_n = 1/n^{1/2}$)
(e) diverges b/c the first part diverges by p-series and the second converges by AST
(f) converges by ACT (+ LCT with $b_n = 1/n^{3/2}$ and p-series)