Power series, Taylor series

1. Determine the *interval* of convergence (i.e. the radius, the center *and* the behaviour at the endpoints).

a) $\sum_{n=0}^{\infty} x^n$ b) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ c) $\sum_{n=0}^{\infty} 3^n x^n$ d) $\sum_{n=0}^{\infty} 2^n x^{3n+1}$

e) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ f) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ g) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$ h) $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{n}$

2. Determine the center and the radius of convergence (not every infinite sum is given in the standard power series form, but each can be rewritten that way).

a) $\sum_{n=1}^{\infty} \frac{3^n}{n!} x^n$ b) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}$ c) $\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n}$ d) $\sum_{n=1}^{\infty} \frac{n^2 (x+1)^n}{3^n}$

e) $\sum_{n=1}^{\infty} \frac{n^2 (x+1)^{2n}}{3^n}$ f) $\sum_{n=1}^{\infty} \frac{n^2 (x+1)^{2n+1}}{3^n}$ g) $\sum_{n=1}^{\infty} \frac{n^2 (2x+1)^{2n}}{3^n}$ h) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

3. Determine the center and the radius of convergence and give a *closed formula (as a function of x)* for the power series sum (valid at least for values of x in the interior of the interval of convergence).

a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ b) $\sum_{n=1}^{\infty} n x^{n-1}$ c) $\sum_{n=2}^{\infty} n x^{n-2}$ d) $\sum_{n=1}^{\infty} \frac{n}{n+1} (x-2)^{n-1}$

e) $\sum_{n=0}^{\infty} (n+1)(n+2) x^n$ f) $\sum_{n=2}^{\infty} (-1)^n \frac{x^{2n-1}}{2n-1}$ g) $\sum_{n=0}^{\infty} \frac{(n+1)}{5^{n+1}} x^n$ h) $\sum_{n=2}^{\infty} \frac{x^n}{n 5^{n-1}}$

4. Determine the Taylor series of the given function $f(x)$ centered at the given value a.

a) $f(x) = \frac{1}{\sqrt[3]{5+x^2}}, a = 0$ b) $f(x) = \frac{1}{x^2 - 3x + 2}, a = 0$ c) $f(x) = \frac{x+1}{x+3}, a = -1$

d) $f(x) = \ln(8 - 3x^2), a = 0$ e) $f(x) = \ln \sqrt{\frac{1+x}{1-x}}, a = 0$ f) $f(x) = \cos x . \sin x, a = 0$

5. Determine the value of the following series.

a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ b) $\sum_{n=1}^{\infty} \frac{1}{n 2^{n+1}}$ c) $\sum_{n=1}^{\infty} \frac{1}{n 5^n}$

d) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$ e) $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ f) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

Answer Key

1. center $a=0$ in all cases
   (a) $R=1$, IoC$=(-1,1)$; geometric series with $r=x$
   (b) $R=2$, IoC$=(-2,2)$; geometric series with $r=x/2$
   (c) $R=1/3$, IoC$=(-1/3,1/3)$; geometric series with $r=3x$
   (d) $R=1/\sqrt[3]{2}$, IOC$=(-1/\sqrt[3]{2},1/\sqrt[3]{2})$; geometric series with $r=2x^3$
   (e) $R=1$, IoC$=[-1,1)$; R as in (a), $x=-1$ converges by Alternating Series Test, $x=1$ diverges by p-Series
   (f) $R=1$, IoC$=[-1,1]$; R as in (a), $x=-1$ converges by AST, $x=1$ converges by p-Series
   (g) $R=1$, IoC$=(-1,1]$; R as in (a), $x=-1$ diverges by p-Series, $x=1$ converges by AST
   (h) $R=1$, IoC$=[-1,1]$; R as without $1/n$, $x=-1,1$ converge by AST

2. (a) $a=0$, $R=\infty$; Ratio Test or compare with e^x series
   (b) $a=3$ and $R=1$; R as for 1a
   (c) $a=1$ and $R=1/2$; R as for $\sum 2^n x^n$
   (d) $a=-1$ and $R=3$; R as for $\sum x^n/3^n$
   (e) $a=-1$ and $R=\sqrt{3}$; R as for $\sum x^{2n}/3^n$
   (f) $a=-1$ and $R=\sqrt{3}$; same as (e)
   (g) $a=-1/2$ and $R=\sqrt{3}/2$; substitute $2x$ for x in (e)
   (h) $a=0$, $R=e$; Ratio Test

3. (a) $a=0$, $R=1$, $-\ln(1-x)$; from $1/(1-x)$ series
   (b) $a=0$, $R=1$, $1/(1-x)^2$; from $1/(1-x)$ series
   (c) $a=0$, $R=1$, $(2-x)/(1-x)^2$; from (b)
   (d) $a=2$, $R=1$, $-1/((2-x)(3-x)) + (\ln(3-x))/(x-2)^2$; from $1/(1-x)$ series and (a)
   (e) $a=0$, $R=1$, $2/(1-x)^3$; from (b)
   (f) $a=0$, $R=1$, $x-\tan^{-1}(x)$; from $1/(1-x)$ series
   (g) $a=0$, $R=5$, $5/(5-x)^2$; from (b)
   (h) $a=0$, $R=5$, $-x-5\ln(1-x/5)$; from (a)

4. (a) $\displaystyle \frac{1}{\sqrt[3]{5}} + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 4\cdot\ldots\cdot(3n-2)}{15^n \sqrt[3]{5}\, n!} x^{2n}$ on $(-\sqrt{5},\sqrt{5})$; from $(1+x)^{-1/3}$
   (b) $\displaystyle \sum_{n=0}^{\infty}\Big(1-\frac{1}{2^{n+1}}\Big)x^n$ on $(-1,1)$; from $1/(1-x)$
   (c) $\displaystyle \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x+1)^n}{2^n}$ on $(-3,1)$; from $1/(1-x)$
   (d) $\displaystyle 3\ln 2 - \sum_{n=1}^{\infty}\frac{3^n}{n\,8^n}x^{2n}$ on $(-2\sqrt{2}/\sqrt{3}, 2\sqrt{2}/\sqrt{3})$; from $1/(1-x)$
   (e) $\displaystyle \sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}$ on $(-1,1)$; from $1/(1-x)$
   (f) $\displaystyle \sum_{n=0}^{\infty}\frac{(-4)^n x^{2n+1}}{(2n+1)!}$ on $(-\infty,\infty)$; from double-angle formula and $\sin(x)$ TS

5. (a) 2; use 3b (b) $\frac{1}{2}\ln 2$; use 3a (c) $\ln(5/4)=\ln 5 - 2\ln 2$; use 3a
   (d) $e^{-2}-1$; use e^x TS (e) $\frac{1}{2}(e^x-e^{-x})-1$; use e^x TS (f) 1; use e^x TS