MAT 127: Calculus C, Spring 2022
Solutions to Midterm II

Problem 1 (20pts)

Answer Only: *no explanation is required. Write your answer to each question in the corresponding box in the simplest possible form. No credit will be awarded if the answer in the box is wrong; partial credit may be awarded if the answer in the box is correct, but not in the simplest possible form. In (a)-(c), assume that the limits exist.*

(a; 5pts) *Find the limit of the sequence* $a_n = \cos\left(\frac{n\pi}{n+\pi}\right)$

$\boxed{-1}$

This is similar to HW6 VI.1,2. Note that

$$a_n = \cos\left(\frac{n\pi/n}{(n+\pi)/n}\right) = \cos\left(\frac{\pi}{n/n+\pi/n}\right) = \cos\left(\frac{\pi}{1+\pi/n}\right).$$

Plugging in $n=\infty$, we obtain

$$a_n \longrightarrow \cos\left(\frac{\pi}{1+\pi/\infty}\right) = \cos\left(\frac{\pi}{1+0}\right) = \cos\pi = -1.$$

Grading: wrong answer 0pts; $\cos\pi$ 3pts; as above 5pts

(b; 5pts) *Find the limit of the sequence* $a_n = \left(1-\frac{3}{n}\right)^{9n}$

$\boxed{\mathrm{e}^{-27}}$

This is similar to HW6 WA 3. Let $b_n = \ln a_n$, so that

$$b_n = 9n\cdot\ln\left(1-\frac{3}{n}\right) = 9\frac{\ln\left(1-3\frac{1}{n}\right)}{1/n}.$$

Replacing $1/n \longrightarrow 0^+$ with $x\longrightarrow 0$ makes sense in this case and

$$\lim_{n\longrightarrow\infty} b_n = 9\lim_{x\longrightarrow 0}\frac{\ln(1-3x)}{x} = 9\lim_{x\longrightarrow 0}\frac{\frac{1}{1-3x}\cdot(-3)}{1} = 9\frac{\frac{1}{1-3\cdot 0}\cdot(-3)}{1} = -27.$$

The above limit computation uses l'Hospital. It is applicable here, since $\ln(1-3x), x \longrightarrow 0$ as $x\longrightarrow 0$ (the top and bottom of a fraction must *both* approach 0 or $\pm\infty$ for l'Hospital to apply). Since $b_n \longrightarrow -27$, $a_n = \mathrm{e}^{b_n} \longrightarrow \mathrm{e}^{-27}$.

Grading: wrong answer 0pts; as above 5pts

$$\boxed{\frac{\sqrt{5}-1}{2}}$$

(c; 5pts) *Find the limit of the sequence recursively defined by*

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{1+a_n} \quad \text{if } n \ge 1$$

This is similar to HW6 WA4 and VI.3, except the sequence is now assumed to be convergent. Then,

$$a = \lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} a_{n+1} = \lim_{n\longrightarrow\infty} \frac{1}{1+a_n} = \frac{1}{1+\lim_{n\longrightarrow\infty} a_n} = \frac{1}{1+a}\,.$$

So, $a=1/(1+a)$ or $a^2+a-1=0$. This gives

$$a = \frac{-1 \pm \sqrt{1^2 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}\,.$$

Since $a_n > 0$ for all n, $a \ge 0$ and so we must take + above.

Grading: wrong answer 0pts; as above (with either order of the terms in the numerator or split into fractions) 5pts; not in the simplest form 4pts

$$\boxed{\frac{58}{55}}$$

(d; 5pts) *Write the number* $1.0\overline{54} = 1.0545454\ldots$ *as a simple fraction*

This is similar to HW7 WA4,5:

$$\begin{aligned} 1.0\overline{54} &= 1 + .054 + .054 \cdot \frac{1}{100} + .054 \cdot \frac{1}{100^2} + \ldots \\ &= 1 + \frac{54/1000}{1-\frac{1}{100}} = 1 + \frac{54/10}{99} = 1 + \frac{3}{55} = \frac{55+3}{55} = \frac{58}{55} \end{aligned}$$

Grading: wrong answer 0pts; as above 5pts; $1\frac{3}{55}$ or not simplified 4pts; both issues 3pts

Problem 2 (15pts)

Suppose $a_1, a_2, a_3, \ldots$ *is a sequence such that* $a_1, a_2, a_3 \ldots \geq 0$ *and the series* $\sum_{n=2}^{\infty} a_n$ *converges. For each question below, circle your answer and justify it below*

(a; 2pts) *Does the sequence* $a_1, a_2, a_3, \ldots$ *converge?* **yes** **no** **impossible to tell**

Since the series $\sum_{n=2}^{\infty} a_n$ converges, the sequence $a_2, a_3, \ldots$ converges (to 0). This is by the *Test for Divergence of Series*. Sticking a_1 at the beginning of a sequence does not affect its convergence.

Grading: wrong answer 0pts regardless of explanation; correct answer 1pt; minimal explanation 1pt (the *sticking* comment not necessary)

(b; 3pts) *Does the series* $\sum_{n=2}^{\infty} \frac{1}{2+a_n}$ *converge?* **yes** **no** **impossible to tell**

Since the sequence $\frac{1}{2+a_n}$ converges to $\frac{1}{2+0} = \frac{1}{2} \neq 0$, the series $\sum_{n=2}^{\infty} \frac{1}{2+a_n}$ diverges. This is by the *Test for Divergence of Series.*

Grading: wrong answer 0pts regardless of explanation; correct answer 1pt; sequence not converging to 0 1pt; minimal explanation for the latter (or converges to 1/2) 1pt

(c; 5pts) *Does the series* $\sum_{n=1}^{\infty} \sqrt{a_n}$ *converge?* **yes** **no** **impossible to tell**

Since $a_n \longrightarrow 0$, $\sqrt{a_n} > a_n$ for all n large. Thus, the convergence of the series $\sum_{n=2}^{\infty} a_n$ says nothing about the convergence of the series $\sum_{n=1}^{\infty} \sqrt{a_n}$. For example, if $a_n = 1/n^4$, then both series converge by the *p-Series Test*. If $a_n = 1/n^2$, then the first series converge and the second diverges by the *p-Series Test.*

Grading: wrong answer 0pts regardless of explanation; correct answer 1pt; explanation with $\sqrt{a_n}$ being larger than a_n 3pts; illustration with examples 4pts (not in addition to the 3pts).

(d; 5pts) *Does the series* $\sum_{n=1}^{\infty} a_n^2$ *converge?* **yes** **no** **impossible to tell**

Since $a_n \longrightarrow 0$, $0 \leq a_n^2 \leq a_n$ for all n large. By the *Comparison Test*, the convergence of the series $\sum_{n=2}^{\infty} a_n$ thus implies the convergence of the "smaller" series $\sum_{n=1}^{\infty} a_n^2$; the extra term a_1 does not matter. One could also use the *Limit Comparison Test*, *after* dropping all $a_n = 0$; these do not effect the convergence of either series.

Grading: wrong answer 0pts regardless of explanation; correct answer 1pt; explanation correct on the substance 3pts; fully correct 4pts (not in addition to the 3pts).

Problem 3 (15pts)

Find all values of p for which the series

$$\sum_{n=1}^{\infty}\left(\frac{n^p}{n^3+6n^2+11n+6}+\frac{5^{n/2}}{3^{n+p}}\right)$$

converges. Write your answer in the box to the right and justify it below.

$\boxed{p<2}$

The series

$$\sum_{n=1}^{\infty}\frac{5^{n/2}}{3^{n+p}}=\sum_{n=1}^{\infty}\frac{\sqrt{5}^n}{3^n3^p}=\frac{1}{3^p}\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}}{3}\right)^n$$

is a geometric series with the ratio $r=\sqrt{5}/3$. Since $|r|<1$ in this case, this series converges no matter what p is. This implies that

$$\sum_{n=1}^{\infty}\left(\frac{n^p}{n^3+6n^2+11n+6}+\frac{5^{n/2}}{3^{n+p}}\right)=\sum_{n=1}^{\infty}\frac{n^p}{n^3+6n^2+11n+6}+\sum_{n=1}^{\infty}\frac{5^{n/2}}{3^{n+p}}$$

and that the series on LHS converges if and only if the first series on RHS converges.

We note that

$$0\le\frac{1}{24}\frac{1}{n^{3-p}}=\frac{n^p}{n^3+6n^3+11n^3+6n^3}\le\frac{n^p}{n^3+6n^2+11n+6}\le\frac{n^p}{n^3}=\frac{1}{n^{3-p}}\,.$$

By the *p-Series Test*, the series $\sum_{n=1}^{\infty}\frac{1}{n^{3-p}}$ converges if $3-p>1$, i.e. if $p<2$. By the *Comparison Test* and the above inequalities, the "smaller" series $\sum_{n=1}^{\infty}\frac{n^p}{n^3+6n^2+11n+6}$ then also converges. By the *p-Series Test*, the series

$$\sum_{n=1}^{\infty}\frac{1}{24}\frac{1}{n^{3-p}}=\frac{1}{24}\sum_{n=1}^{\infty}\frac{1}{n^{3-p}}$$

diverges if $3-p\le1$, i.e. if $p\ge2$. By the *Comparison Test* and the above inequalities, the "larger" series $\sum_{n=1}^{\infty}\frac{n^p}{n^3+6n^2+11n+6}$ then also diverges. Thus, the series in the statement of the problem converges if and only if $p<2$.

We can also use the *Limit Comparison* (or *Looks Like*) *Test* to study the convergence of the series $\sum_{n=1}^{\infty}\frac{n^p}{n^3+6n^2+11n+6}$. The terms in this series are always positive and look like n^p/n^3 (n^3 completely dominates n^2, etc. as as $n\longrightarrow\infty$). We verify this by computing

$$\begin{aligned}\lim_{n\longrightarrow\infty}\frac{n^p/(n^3+6n^2+11n+6)}{n^p/n^3}&=\lim_{n\longrightarrow\infty}\frac{n^3}{n^3+6n^2+11n+6}=\lim_{n\longrightarrow\infty}\frac{n^3/n^3}{(n^3+6n^2+11n+6)/n^3}\\&=\lim_{n\longrightarrow\infty}\frac{1}{1+6/n+11/n^2+6/n^3}=\frac{1}{1+6/\infty+11/\infty+6/\infty}=1\,.\end{aligned}$$

Thus, the series $\sum_{n=1}^{\infty}\frac{n^p}{n^3+6n^2+11n+6}$ converges if and only if the series

$$\sum_{n=1}^{\infty}\frac{n^p}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^{3-p}}$$

converges. By the *p-Series Test*, the last series converges if and only if $3-p>1$, i.e. $p<2$.

Grading: correct answer 3pts; explanation of convergence of geometric series 4pts; statement of equivalence of convergence of the original series and its first part as a consequence of this 1pt (this need not be completely direct, but should be clearly implied in the right context); proper use of Comparison or Limit Comparison Test and justification 5pts; proper use of p-Series Test 2pts

(bonus 10pts) *Pick any value of p for which the above series converges and find the sum of the resulting series explicitly.*

Take $p=0$. We then compute

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^3+6n^2+11n+6}+\frac{5^{n/2}}{3^n}\right)=\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}+\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}}{3}\right)^n.$$

The sum of the last series is given by

$$\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}}{3}\right)^n=\frac{\sqrt{5}/3}{1-\sqrt{5}/3}=\frac{\sqrt{5}}{3-\sqrt{5}}=\frac{\sqrt{5}(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{3\sqrt{5}+5}{3^2-5}=\frac{3\sqrt{5}+5}{4}$$

In order to compute the other sum, we use quick partial fractions twice. To keep things symmetric, we first split $(n+1)(n+3)$:

$$\frac{1}{(n+1)(n+3)}=\frac{1}{+3-(+1)}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)=\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+3}\right).$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}-\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+3)}\right).$$

We now apply quick partial fractions to the terms in the first series on RHS:

$$\frac{1}{(n+1)(n+2)}=\frac{1}{+2-(+1)}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\frac{1}{n+1}-\frac{1}{n+2}$$

The sequence of partial sums for the first series above is thus given by

$$s_n=\sum_{k=1}^{k=n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$
$$=\frac{1}{2}-\frac{1}{n+2}.$$

By the definition of the sum of a series, this implies that

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}=\lim_{n\longrightarrow\infty}s_n=\frac{1}{2}-\frac{1}{\infty+2}=\frac{1}{2}.$$

From this, we also find that

$$\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+3)}=\sum_{k=2}^{\infty}\frac{1}{(k+1)(k+2)}=\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}-\frac{1}{(1+1)(1+2)}=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}.$$

The first equality above is obtained by replacing $n+1$ (with $n \geq 1$) by $k \geq 2$; the second is obtained by adding the $k=1$ term back into the series and subtracting it off outside of the series. We can also compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$ similarly to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$. In this case, $s_n = \frac{1}{3} - \frac{1}{n+3}$.

Putting everything together, we obtain

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^3+6n^2+11n+6} + \frac{5^{n/2}}{3^n} \right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{3\sqrt{5}+5}{4} = \frac{1}{12} + \frac{3\sqrt{5}+5}{4} = \boxed{\frac{16+9\sqrt{5}}{12}}$$

The sum of the series can be similarly computed for $p = -1, 1, 2$, but the computation becomes more difficult (especially for $p = -1$).

Grading: sum of geometric series fully simplified and with p chosen 2pts (no square roots in denominator); partial fractions 4pts (with reductions for computational errors); pairwise cancellation done properly 2pts; rest 2pts; reduction if the answer is not fully simplified

Problem 4 (10pts)

Determine the sequence s_n of partial sums (sum of the first n terms) corresponding to the series

$$\sum_{n=1}^{\infty}(-1)^n\left(\cos\left(\frac{\pi}{2n}\right)-\cos\left(\frac{\pi}{2(n+2)}\right)\right).$$

Does this series converge? If so, what is its sum? Justify your answers.

The sequence of partial sums is given by

$$\begin{aligned}
s_n &= \sum_{k=1}^{k=n}(-1)^k\left(\cos\left(\frac{\pi}{2k}\right)-\cos\left(\frac{\pi}{2(k+2)}\right)\right)\\
&= -\left(\cos\left(\frac{\pi}{2\cdot 1}\right)-\cos\left(\tfrac{\pi}{2\cdot 3}\right)\right)+\left(\cos\left(\frac{\pi}{2\cdot 2}\right)-\cos\left(\tfrac{\pi}{2\cdot 4}\right)\right)\\
&\quad -\left(\cos\left(\tfrac{\pi}{2\cdot 3}\right)-\cos\left(\frac{\pi}{2\cdot 5}\right)\right)+\left(\cos\left(\tfrac{\pi}{2\cdot 4}\right)-\cos\left(\frac{\pi}{2\cdot 6}\right)\right)\\
&\quad \ldots+(-1)^n\left(\cos\left(\frac{\pi}{2n}\right)-\cos\left(\frac{\pi}{2(n+2)}\right)\right)\\
&= -\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{4}\right)-(-1)^{n-1}\cos\left(\frac{\pi}{2(n+1)}\right)-(-1)^n\cos\left(\frac{\pi}{2(n+2)}\right)\\
&= \boxed{\frac{\sqrt{2}}{2}-(-1)^{n-1}\cos\left(\frac{\pi}{2(n+1)}\right)-(-1)^n\cos\left(\frac{\pi}{2(n+2)}\right).}
\end{aligned}$$

The second term in each pair with $k\le n-2$ gets canceled by the first term in the pair $k+2$. This leaves the first terms in the first two pairs and the second terms in the last two pairs. While this reasoning does not directly apply to s_1, the above formula is valid for all $n\ge 1$.

We note that

$$\begin{aligned}
\lim_{n\longrightarrow\infty}\left(\cos\left(\frac{\pi}{2(n+1)}\right)-\cos\left(\frac{\pi}{2(n+2)}\right)\right) &= \lim_{n\longrightarrow\infty}\cos\left(\frac{\pi}{2(n+1)}\right)-\lim_{n\longrightarrow\infty}\cos\left(\frac{\pi}{2(n+2)}\right)\\
&= \cos\left(\frac{\pi}{2(\infty+1)}\right)-\cos\left(\frac{\pi}{2(\infty+2)}\right)=\cos 0-\cos 0=0.
\end{aligned}$$

By the *Squeeze Theorem for Sequences*, this implies that

$$\lim_{n\longrightarrow\infty}\left(-(-1)^{n-1}\cos\left(\frac{\pi}{2(n+1)}\right)-(-1)^n\cos\left(\frac{\pi}{2(n+2)}\right)\right)=\lim_{n\longrightarrow\infty}(-1)^n\left(\cos\left(\frac{\pi}{2(n+1)}\right)-\cos\left(\frac{\pi}{2(n+2)}\right)\right)=0.$$

Therefore, the sequence s_n of partial sums converges to

$$\lim_{n\longrightarrow\infty}s_n=\frac{\sqrt{2}}{2}+0\,.$$

This means that the original series $\boxed{\text{converges}}$ and its sum is

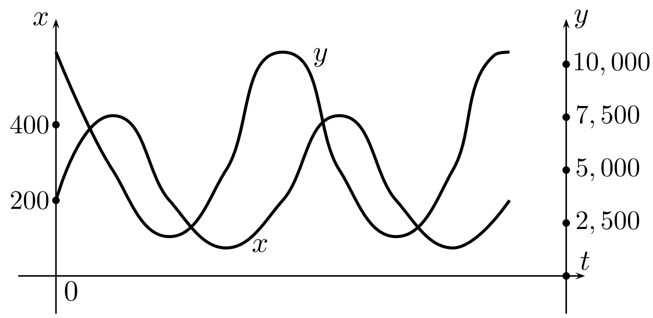
$$\sum_{n=1}^{\infty}(-1)^n\left(\cos\left(\frac{\pi}{2n}\right)-\cos\left(\frac{\pi}{2(n+2)}\right)\right)=\boxed{\frac{\sqrt{2}}{2}}$$

Grading: definition of s_n or the right setup for computing it 1pt; clear indication or statement of two-step cancellation and simplifying to final answer for s_n 3pts ($1/\sqrt{2}$ is fine here); converges 1pt; converge/sum for series equivalent to same for s_n 1pt each; justification of convergence of s_n 3pts (the tail terms do not approach 0 separately)

Problem 5 (20pts)

A two-species interaction is modeled by the system of differential equations below, with t denoting time.

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{10}x + \frac{1}{50{,}000}xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{1}{200}xy \end{cases}$$



(a; 3pts) *Which of the following best describes the interaction modeled by this system?*

(i) *predator-prey* (ii) *competition for same resources* (iii) *cooperation for mutual benefit*

Circle your answer above and justify it below.

Because of the coefficient of $+\frac{1}{50{,}000}$ in front of xy in the first equation, the x-species benefits from the presence of the y-species (the growth rate of the former is increased if the population of the latter is nonzero). Because of the coefficient of $-\frac{1}{200}$ in front of xy in the second equation, the y-species is hurt by the presence of the x-species. The x-species is thus the predator, and the y-species is the prey (given the above three choices).

Grading: wrong answer 0pts regardless of explanation; correct answer 2pts; two-part justification 1pt (anything in parenthesis not required)

(b; 7pts) *Find the equilibrium (constant) solutions of the system and explain their significance relative to the interaction the system is modeling.* **Answer Only:** *clearly write down each equilibrium solution followed by its significance below, with one of these statements per line. Use scrap paper or the back side of a page in the exam to work out your answer.*

(0,0):	no predators or prey ever
(200,5000):	5,000 prey are precisely enough to support 200 predators and be contained by them

We need to find pairs of numbers (x, y) such that

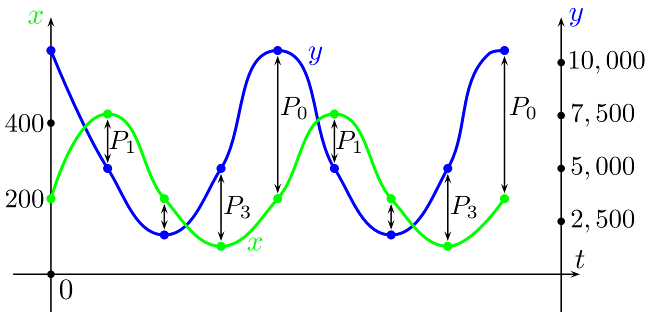
$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \end{cases} \iff \begin{cases} -\frac{x}{10}\left(1 - \frac{1}{5{,}000}y\right) = 0 \\ y\left(1 - \frac{1}{200}x\right) = 0 \end{cases} \iff \begin{cases} x = 0 \text{ or } y = 5{,}000 \\ y = 0 \text{ or } x = 200 \end{cases}$$

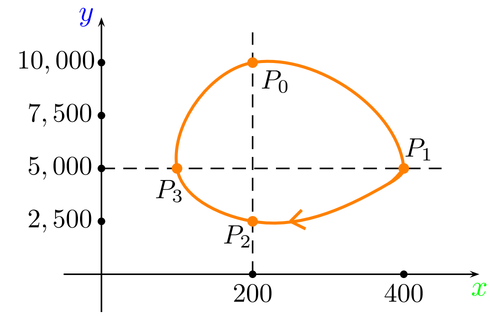
We must consider all possible cases of taking one condition from the first line in the last expression above and one condition from the second line. This gives 4 possibilities:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x = 0 \\ x = 200 \end{cases} \qquad \begin{cases} y = 5{,}000 \\ y = 0 \end{cases} \qquad \begin{cases} y = 5{,}000 \\ x = 200 \end{cases}$$

The second and third systems of equations have no solutions, while the first and the fourth give us $(x, y) = (0, 0)$ and $(x, y) = (200, 5000)$, respectively.

Grading: 1 correct pair 2pts, 2 correct 5pts; 3pts reduction for each additional pair (without going below 0); reasonable significance 1pt each (c; 10pts) *The diagram above shows the graphs of functions $x=x(t)$ and $y=y(t)$ so that the pair (x,y) solves the above system of differential equations. Sketch the corresponding (directed) phase trajectory below, indicating coordinates of whatever points possible. Explain/indicate how you make your sketch!*





Begin by copying the scale labels on the x-axis and y-axis from the left diagram to the right diagram (just 400 and 200 in the first case). At time $t=0$, the x and y-populations are 200 and about 10000, respectively, giving the starting point $P_0 \approx (200, 10000)$ in the phase plane. The first interesting feature in the two graphs is the peak in the x-graph (corresponding to the right-most point in the phase trajectory); at this time, the x and y-populations are about 400 and 5000 respectively, giving the point $P_1 \approx (400, 5000)$ in the phase plane. The second interesting feature in the two graphs is the sag in the y-graph (corresponding to the lowest point in the phase trajectory); at this time, the x and y-populations are 200 and about 2500, respectively, giving the point $P_2 \approx (200, 2500)$ in the phase plane. The third interesting feature in the two graphs is the sag in the x-graph (corresponding to the left-most point in the phase trajectory); at this time, the x and y-populations are about 100 and 5000 respectively, giving the point $P_1 \approx (100, 5000)$ in the phase plane. The next interesting feature in the two graphs is the peak in the y-graph (corresponding to the lowest point in the phase trajectory); at this time, the x and y-populations are 200 and about 10000, respectively, giving the point P_0 in the phase plane again.

After that, the points repeat *periodically*. The x-values of 200 and the y-values of 5000 above are exact because they correspond to the x- and y-coordinates of the nonzero equilibrium in (b). The rotation in this case is clockwise because the predator is on the horizontal axis.

Grading: x- and y-axes properly marked 1pt; x- and y-axes properly scaled 1pt; correct general shape, direction, and relation with the equilibrium 5pts; correct starting point, coordinates of the points, and/or justification up to 3pts