MAT 127: Calculus C, Fall 2009
Solutions to Midterm II

Problem 1 (15pts)

Determine whether each of the following sequences converges or diverges; justify your answer. You do not have to determine the limit if the sequence converges.

(a; 5pts) $a_n = \dfrac{(-1)^n n^4}{3n^4+1}$ $\boxed{\text{diverges}}$ because

$$a_n = \frac{(-1)^n n^4}{3n^4+1} = (-1)^n \frac{n^4/n^4}{3n^4/n^4 + 1/n^4} = (-1)^n \frac{1}{3+1/n^4}\,;$$

as $n \longrightarrow \infty$ the fraction approaches $1/3$, while the entire expression keeps on jumping from near $1/3$ (even n) to near $-1/3$ (odd n).

Grading: correct answer 2pts; dividing by n^4, obtaining $1/3$ as limit of the fraction, and some comment on the sign 1pt each (-1pt for *mce*); wrong answer with complete explanation containing *mce* in division by n^4 or forgetting $(-1)^n$ max 2pts.

mce = minor computational error

(b; 5pts) $a_n = 1 + \dfrac{\sin(n\pi/2)\ln n}{n}$ $\boxed{\text{converges}}$ because
$(\ln n)/n$ approaches 0, while $|\sin(n\pi/2)| \le 1$; so $\sin(n\pi/2)(\ln n)/n \longrightarrow 0$ and $a_n \longrightarrow 1$.

Grading: correct answer 2pts; correct statements regarding $(\ln n)/n$ and $\sin(n\pi/2)$ 2pts and 1pt, respectively; *so* ... not required; wrong answer with complete explanation misidentifying $\lim_{n\longrightarrow\infty}(\ln n)/n$ max 2pts.

(c; 5pts) $a_1 = 1, \quad a_{n+1} = \dfrac{a_n+9}{2}$ $\boxed{\text{converges}}$ because

- $a_n \le 9$ for all n: this is true for $n=1$; if this is true for n, $a_{n+1} = (a_n+9)/2 \le (9+9)/2 = 9$ and so this is true for $n+1$;
- $a_n \le a_{n+1}$: $a_{n+1} = (a_n+9)/2 \ge (a_n + a_n)/2 = a_n$.

So $\{a_n\}$ converges by the Monotonic Sequence Theorem.

Grading: correct answer 2pts; *bounded above* 2pts; *increasing* 1pt; last sentence not required; wrong answer 0pts regardless of explanation.

Alternatively (bonus 2pts, whether or not in addition to a standard argument): the sequence $\{a_n\}$ is obtained by starting with some $a_1 < 9$, then taking a_2 to be the mid-point of the segment $[a_1, 9]$, a_3 to be the mid-point of the segment $[a_2, 9]$, and so on. This sequence of consecutive mid-points approaches 9, since at each step the new distance to 9 is reduced to $1/2$ of the old distance to 9.

Grading: the substance must be identical to the above for full credit, but could be worded (with a bit of care) in terms of the sequence $b_n = 9 - a_n$; 2nd half of last sentence not required.

Problem 2 (20pts)

Each of the following sequences converges (you do not need to check this); determine its limit (show your work!).

(a; 6pts) $a_n = \dfrac{n}{1+\sqrt{4n^2+1}}$

$$a_n = \frac{n/n}{1/n+\sqrt{4n^2+1}/n} = \frac{1}{1/n+\sqrt{4n^2/n^2+1/n^2}} = \frac{1}{1/n+\sqrt{4+1/n^2}}$$

So $a_n \longrightarrow 1/(0+\sqrt{4+0}) = \boxed{1/2}$

Grading: correct answer 3pts; dividing by n 1pt; 2pts to the answer; *mce* -1pt (from 6pts); taking n under $\sqrt{\ }$ without squaring it resulting in limit of 0 -2pts (from 6pts)

mce = minor computational error

(b; 7pts) $a_n = \dfrac{5^n}{n!}$ (reminder: $n! = 1\cdot 2\cdot \ldots \cdot n$)

1st Approach. Since the limit exists and $a_{n+1} = \dfrac{5}{n+1}a_n$,

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} a_{n+1} = \lim_{n\longrightarrow\infty}\left(\frac{5}{n+1}a_n\right) = \left(\lim_{n\longrightarrow\infty}\frac{5}{n+1}\right)\cdot\left(\lim_{n\longrightarrow\infty} a_n\right) = 0\cdot\left(\lim_{n\longrightarrow\infty} a_n\right) = \boxed{0}$$

ok without $n\longrightarrow\infty$

2nd Approach. $a_n \geq 0$ for all n and

$$a_{5+n} = \frac{5^5}{5!}\cdot\frac{5}{6}\cdot\frac{5}{7}\cdot\ldots\cdot\frac{5}{5+n} \leq a_5\cdot\left(\frac{5}{6}\right)^n.$$

Since the geometric sequence $a_5(5/6)^n$ converges to 0 (as does the constant sequence $\{0\}$), $\{a_n\}$ also converges to $\boxed{0}$

3rd Approach. Since $a_n > 0$ and $\lim_{n\longrightarrow 0}\dfrac{a_{n+1}}{a_n} = \lim_{n\longrightarrow 0}\dfrac{5}{n+1} = 0$, the series $\sum_{n=1}^{\infty} a_n$ converges and thus $a_n \longrightarrow \boxed{0}$

Grading: correct answer 3pts; in 1st approach, recursion statement and its application 2pts each (*exists* not required); in 2nd approach, lower and upper bounds 2pts each (*as does* not required); in 3rd approach, 3pts for series convergence statement and 1pt for sequence convergence conclusion; wrong answer 0pts regardless of explanation.

(c; 7pts) $\sqrt{6},\ \sqrt{6+\sqrt{6}},\ \sqrt{6+\sqrt{6+\sqrt{6}}},\ \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}},\ \ldots$

This sequence satisfies $a_{n+1} = \sqrt{6+a_n}$. Thus, if a is its limit, $a = \sqrt{6+a}$. So $a^2 - a - 6 = 0$; this gives $a = 3, -2$. Since a is non-negative, $a = \boxed{3}$

Grading: correct answer 3pts; recursion statement 2pts; following limit statement 2pts; -1pt (from 7pts) for each *mce*; non-positive answer or 2 answers (e.g. 3 and -2) 0pts regardless of explanation.

Problem 3 (20pts)

Determine whether each of the following series converges or diverges; justify your answer. You do not have to compute the sum if the series converges.

(a; 6pts) $\sum_{n=1}^{\infty}(-1)^n$ $\boxed{\text{diverges}}$ because

(1) the sequence of partial sums $s_n = \sum_{k=1}^{k=n}(-1)^k$ alternates between -1 and 0 and thus diverges;

(2) the terms $(-1)^n$ alternate between -1 and 1 and thus do not converge (to 0);

(3) this is a geometric series with (common ratio) $r=-1$ and thus does not converge.

Grading: correct answer 2pts; either explanation 4pts; minor misstatements or omissions in explanations -1pt each; *partial sums*, *to 0*, and *common ratio* not required; wrong answer 0pts regardless of explanation.

(b; 7pts) $\sum_{n=1}^{\infty}\frac{1}{3^n-2^n}$ $\boxed{\text{converges}}$ because $1/(3^n-2^n)>0$ and

(1) $\lim_{n\longrightarrow\infty}\frac{1/(3^n-2^n)}{1/3^n} = \lim_{n\longrightarrow\infty}\frac{1}{1-(2/3)^n} = 1$ and $\sum_{n=1}^{\infty}\frac{1}{3^n}$ converges being geometric series with $|r|<1$;

(1) $\lim_{n\longrightarrow\infty}\frac{1/(3^n-2^n)}{1/2^n} = \lim_{n\longrightarrow\infty}\frac{(2/3)^n}{1-(2/3)^n} = 0$ and $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges being geometric series with $|r|<1$;

ok without $n\longrightarrow\infty$

(3) $1/(3^n-2^n) = 1/2^n(1.5^n-1) \le 2/2^n$ and $\sum_{n=1}^{\infty}\frac{2}{2^n}$ converges being geometric series with $|r|<1$.

Grading: correct answer 2pts; either explanation 5pts; missing any of the condition checks -1pt each; wrong answer 0pts regardless of explanation.

(c; 7pts) $\sum_{n=1}^{\infty}\tan^2(1/n)$ $\boxed{\text{converges}}$ because

- $\tan^2(1/n)>0$, $\lim_{n\longrightarrow\infty}\frac{\tan^2(1/n)}{(1/n)^2} = \lim_{x\longrightarrow 0}\frac{\sin^2 x}{x^2} = 1$, and $\sum_{n_1}^{\infty}\frac{1}{n^2}$ converges;
- $\tan^2(1/x)$ is positive, continuous, and decreasing on $[1,\infty)$ and $\int_1^{\infty}\tan^2(1/x)\mathrm{d}x = \int_0^1\frac{\tan^2(u)}{u^2}\mathrm{d}u$; the second integral is finite because $\lim_{u\longrightarrow 0}(\tan u)/u = \lim_{u\longrightarrow 0}(\sin u)/u = 1$.

Grading: correct answer 2pts; either explanation 5pts, with $(\sin x)/x$ limit statement 2pts; -1pt for each minor omission or misstatement; wrong answer with complete explanation max 2pts

Problem 4 (20pts)

(a; 10pts) *Write the number $2.0\bar{7} = 2.077777\ldots$ as a simple fraction (p/q for some integers p, q).*

$$2.0\bar{7} = 2 + .07 + \frac{.07}{10} + \frac{.07}{10^2} + \ldots = 2 + \frac{7/100}{1-\frac{1}{10}} = 2 + \frac{7}{90} = \boxed{\frac{187}{90}}$$

Grading: 4pts for setup (1st equality), -1pt for *mce* such as .7 instead of .07; 4pts for sum of geometric series, -1pt for incorrect numerator (based on the setup), -2pts for incorrect denominator; 2pts to simplify the fraction and add in 2; -1pt for $2\frac{7}{90}$ (does not quite answer the question)

mce = minor computational error

(b; 10pts) *Determine the sum of the following convergent series:*

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-9} = -\frac{1}{5} + \frac{1}{7} + \frac{1}{27} + \frac{1}{55} + \frac{1}{91} + \ldots$$

Hint: partial fractions

$$\frac{1}{4n^2-9} = \frac{1}{(2n-3)(2n+3)} = \frac{1}{+3-(-3)}\left(\frac{1}{2n-3} - \frac{1}{2n+3}\right) = \frac{1}{6}\left(\frac{1}{2n-3} - \frac{1}{2n+3}\right)$$

This gives

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-9} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{2n-3} - \frac{1}{2n+3}\right) = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{2n-3} - \frac{1}{2(n+3)-3}\right).$$

Since $1/(2n \pm 3) \longrightarrow 0$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-9} = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{2n-3} - \frac{1}{2(n+3)-3}\right) = \frac{1}{6}\sum_{n=1}^{n=3}\frac{1}{2n-3} = \frac{1}{6}\left(\frac{1}{-1} + \frac{1}{1} + \frac{1}{3}\right) = \boxed{\frac{1}{18}}$$

Alternatively, start writing out the terms in the series to see that the second term in each pair gets canceled by the first term in the pair three steps later, leaving the first terms in the first three pairs.

You can also look at the sequence of partial sums. For $n \geq 3$,

$$\begin{aligned} s_n &= \frac{1}{6}\sum_{k=1}^{k=n}\left(\frac{1}{2k-3} - \frac{1}{2k+3}\right) \\ &= \frac{1}{6}\left(\left(\frac{1}{-1} - \frac{1}{5}\right) + \left(\frac{1}{1} - \frac{1}{7}\right) + \left(\frac{1}{3} - \frac{1}{9}\right) + \left(\frac{1}{5} - \frac{1}{11}\right) + \ldots \left(\frac{1}{2n-3} - \frac{1}{2n+3}\right)\right) \\ &= \frac{1}{6}\left(\frac{1}{-1} + \frac{1}{1} + \frac{1}{3} - \frac{1}{2n-1} - \frac{1}{2n+1} - \frac{1}{2n+3}\right), \end{aligned}$$

since the second term in each pair, except for the last three pairs, is canceled by the first term in the pair three steps later; this leaves only the first terms in the first three pairs and the second terms in the last three pairs. Since $s_n \longrightarrow 1/18$, the sum of the infinite series is $\boxed{1/18}$

Grading: *PFs* correct 4pts (-1pt each if 1/6 is wrong or the overall sign is reversed; -2pts if fractions have the same sign; -2pts for wrong denominators); clear indication/statement of three-step cancellation 4pts (use of $\sum_{n=1}^{n=3}$ counts); $1/(2n-3) \longrightarrow 0$ or equivalent in first 2 approaches or $s_n \longrightarrow 1/18$ in 3rd approach 1pt; rest of computation 1pt; -3pts if not precisely first 3 terms are summed.

Problem 5 (25pts)

The populations of bears and salmon in Salmon River Valley are modeled by the following system of differential equations

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{1}{100}xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{1}{50}y^2 + \frac{1}{1000}xy \end{cases} \qquad (x,y) = (x(t), y(t)),$$

where t denotes time. While the bears can live off berries and other plants, they also feed on the salmon in the river. The salmon has no other predators in the valley.

(a; 5pts) *Does the function $x = x(t)$ above represent the population of the bears or of the salmon? Explain why.* $\boxed{\text{salmon}}$ because of *either* of the following:

- the growth rate of x is reduced by the presence of y, due to the negative coefficient in front of xy in the first equation;
- the growth rate of y is increased by the presence of x, due to the positive coefficient in front of xy in the second equation.

Grading: correct answer 2pts; correct explanation 3pts, -1pt for minor omission or misstatement; both explanations fine; wrong answer 0pts regardless of explanation.

(b; 10pts) *Find all equilibrium (constant) solutions of the system (show your work!) and explain their significance.*
We need to find pairs of numbers (x,y) such that

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \end{cases} \iff \begin{cases} x - \frac{1}{100}xy = 0 \\ y - \frac{1}{50}y^2 + \frac{1}{1000}xy = 0 \end{cases} \iff \begin{cases} x(1 - \frac{1}{100}y) = 0 \\ y(1 - \frac{1}{50}y + \frac{1}{1000}x) = 0 \end{cases}$$

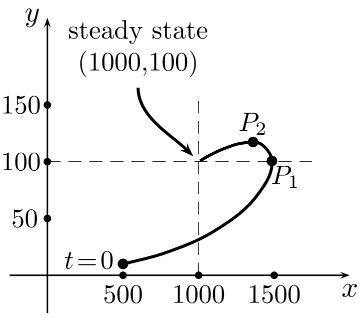
$$\iff \begin{cases} x = 0 \text{ or } y = 100 \\ y = 0 \text{ or } 1 + \frac{1}{1000}x = \frac{1}{50}y \end{cases}$$

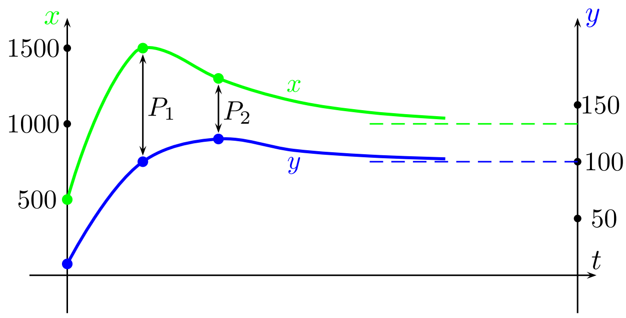
We must consider all possible cases of taking one condition from the first line in the last expression above and one condition from the second line. If $x = 0$ is chosen from the first line, we can take either the first condition from the second line, so $y = 0$, or the second, so $y = 50$. If $y = 100$ is chosen from first line, we are forced to take the second condition from the second line, so $x = 1000$. Thus, there are three equilibrium points $\boxed{(0,0), (0,50), (1000,100)}$ The significance of these is:

(0,0):	no salmon or bears ever
(0,50):	in the absence of salmon, the carrying capacity of the "berries and other plants" is 50 bears
(1000,100):	1000 salmon are precisely enough to supplement the "berries and other plants" to support 100 bears and be contained by them

Grading: the 4th system of conditions on (x,y) is 4pts if at least one of the previous three systems is present or a statement analogous to the 1st system is made (only 3pts otherwise); stopping at an earlier system of conditions is 1pt less for each step not made (so stopping at the 2nd system is 2pts); *each* correct equilibrium point is 1pt; *each* extra "equilibrium pt", such as (0,100) or (-1000,0), is -1pt; reasonable variations of the above significance statements are 1pt each.

(c; 10pts) *The left diagram below*[1] *shows the phase trajectory for the above system of differential equations that starts at (500,10) at* $t = 0$. *Sketch the corresponding graphs of* $x = x(t)$ *and* $y = y(t)$ *as functions of time on the right diagram below*[2]*; add the appropriate markings to the axes. Explain how you make your sketch!*





The starting point of the trajectory is $P_0 = (500, 10)$. Mark the turning points of the trajectory and estimate their coordinates: $P_1 \approx (1500, 100)$ and $P_2 \approx (1300, 120)$; the y-coordinate of P_1 is exact according to the $\frac{\mathrm{d}x}{\mathrm{d}t}$ equation. The limiting point is $P_\infty = (1000, 100)$; this is also exact according to the system of differential equations and (b).

Next copy the markings on x- and y-axes on the left diagram to the right diagram. Mark the asymptotes of both functions: $x = 1000$ and $y = 100$. Above $t_0 = 0$, mark $x = 500$ and $y = 10$ for P_0, keeping track of which is which. Above some t-value, mark $x = 1500$ and $y = 100$ for P_1; above a higher t-value, mark $x = 1300$ and $y = 120$ for P_2. Now draw a smooth curve through the x-points which is either rising or falling between any two adjacent points and descends to toward the x-asymptote; do the same for the y-points. Finally, label the x- and y-graphs. There should be no markings on the t-axis, except possibly for $t = 0$, since the time scale cannot be determined from the phase trajectory.

Grading: x- and y-axes properly marked 1pt; correct starting values 2pts (if x- and y-axes are not properly marked, likely loss of 3pts); next feature x-peak with $y = 100$ 2pts; followed by y-peak 1pt; correct asymptotes and direction of approach 2pts; some explanation 2pts (e.g. points marked on the left diagram and their coordinates indicated); -2pts if the graphs are not labeled or labeled incorrectly (-1pt if there are other clear indications of which graph is which; the graphs are considered labeled if done in different colors and the x- and y-labels on the axes are changed to the same colors).

[1]the original left diagram did not contain the dots marked P_1 and P_2 or the steady state label

[2]the original right diagram consisted of the three axes and labels t, x, y without color