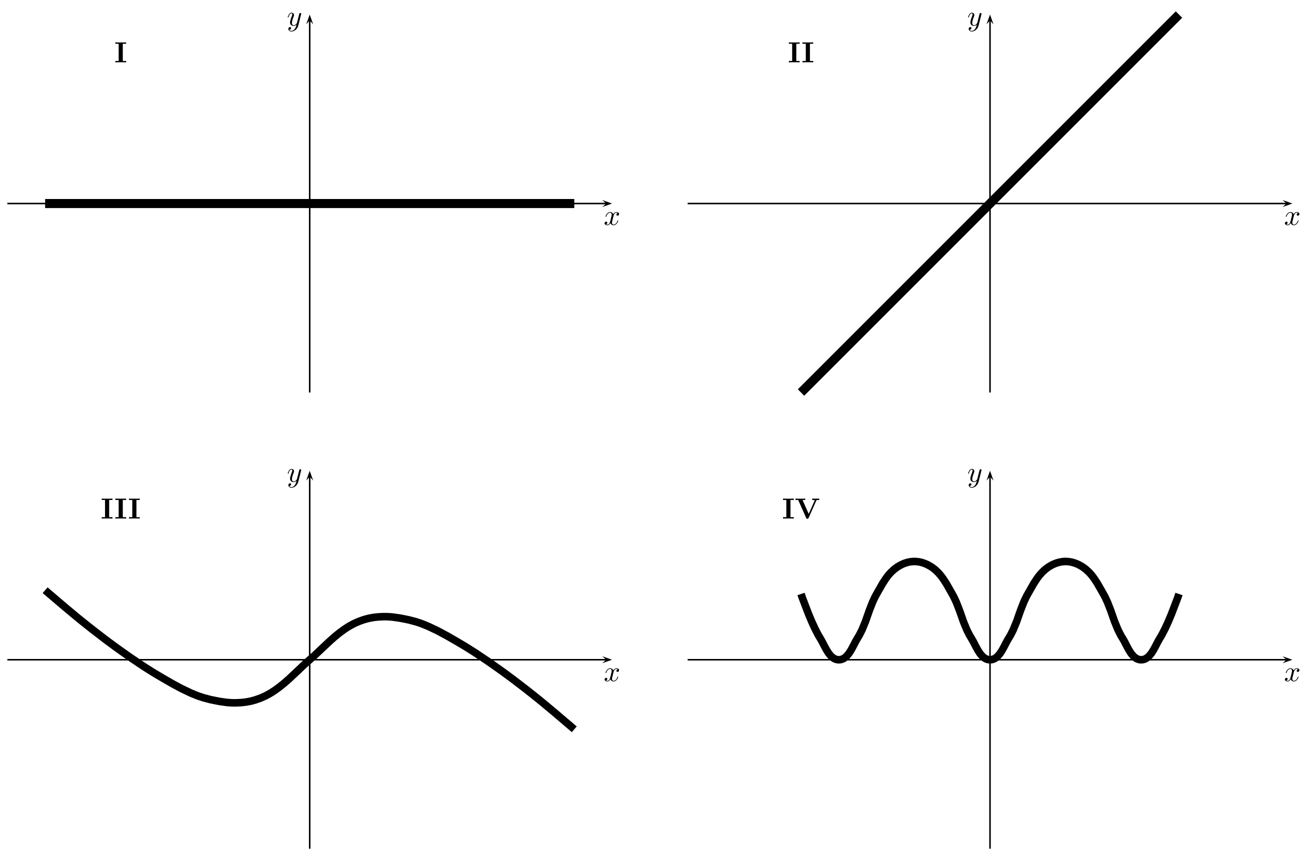
MAT 127: Calculus C, Spring 2022
Solutions to Midterm I

Problem 1 (10pts)

Consider the four differential equations for $y = y(x)$:

(a) $y' = (\cos x)(\sin y)$ (b) $y' = (\sin x)(\cos y)$ (c) $y' = \cos(x-y)$ (d) $y' = \cos(x+y)$.

Each of the four diagrams below shows a solution curve for one of these equations:



Match each of the diagrams to the corresponding differential equation (the match is one-to-one):

diagram	I	II	III	IV
equation	a	c	d	b

Answer Only: *no explanation is required.*

Explanation is on the next page.

Of the four diagrams, the most distinctive is **I**; it shows the graph of the constant function $y(x)=0$. Plugging in this function into the four equations, we get

$$\text{(a)}\ \ 0 \stackrel{?}{=} (\cos x)(\sin 0) \qquad \text{(b)}\ \ 0 \stackrel{?}{=} (\sin x)(\cos 0) \qquad \text{(c)}\ \ 0 \stackrel{?}{=} \cos(x-0) \qquad \text{(d)}\ \ 0 \stackrel{?}{=} \cos(x+0)\,.$$

Of these four potential equalities, only (a) is satisfied for all x; so diagram **I** corresponds to (a).

Of the three remaining graphs, the most distinctive is **IV**; the slope of the graph at $(0,0)$ there is 0. This is the case for the slope of equation (b) at (0,0) because

$$(\sin 0)(\cos 0) = 0 \cdot 1 = 0,$$

but the slopes for the other two remaining equations are $\cos 0 = 1$. So diagram **IV** corresponds to (b).

The two remaining equations have slopes $\cos 0=1$ at $(0,0)$. Thus, the line depicted in **II** must be the graph of the function $y=x$. Plugging in this function into the two remaining equations, we get

$$\text{(c)}\ \ 1 \stackrel{?}{=} \cos(x-x) \qquad \text{(d)}\ \ 1 \stackrel{?}{=} \cos(x+x)\,.$$

Of these two potential equalities, only (c) is satisfied for all x. Diagram **II** thus corresponds to (c), while diagram **III** corresponds to (d).

Grading:

correct − repeats	0-	1	2	3	4
points	0	2	5	9	10

Problem 2 (15pts)

A radioactive substance, Strontium-90, has a half-life of 30 years. A sample of Strontium-90 initially contains 400 mg.

(a; 10pts) *How much Strontium-90 remains after t years?*

Let $y(t)$ be the amount of Strontium-90 in mg remaining after t years; so $y(0) = 400$ mg. Since the decay rate of y is proportional to y, $y(t)$ satisfies the exponential decay equation:

$$y(t) = y(0)\mathrm{e}^{rt} = 400\mathrm{e}^{rt},$$

where $r<0$ is the relative decay rate. Since the half-line of Strontium-90 is 30 years,

$$y(30) = 400\mathrm{e}^{r\cdot 30} = \frac{1}{2}400 \iff \mathrm{e}^{r\cdot 30} = \frac{1}{2} \iff 30r = \ln\frac{1}{2} \iff r = \frac{1}{30}(\ln 1/2) = -\frac{1}{30}\ln 2.$$

Thus, $\boxed{y(t) = 400\mathrm{e}^{-(\ln 2)t/40} = 400\cdot 2^{-t/30} \text{ mg}}$

Grading: $y(t)=400\mathrm{e}^{rt}$ 3pts; $\mathrm{e}^{30r}=1/2$ 2pts; finding r 2pts; final answer 3pts; 2pts off if the units are missing in the final answer; penalty for the answer not in a simplified form, e.g. with $\ln(1/2)$. Either form of the solution in the box is acceptable; the solution could also be carried out with r replaced by $-r$. Finding r is not necessary for full credit (something like $\mathrm{e}^{rt}=(\mathrm{e}^{30r})^{t/30}$ would do instead). Correct answer with no explanation is 4pts.

(b; 5pts) *How long will take for the amount of Strontium-90 to decay to 25 mg?*

We need to find t so that

$$\begin{aligned} y(t) = 400\mathrm{e}^{-(\ln 2)t/30} = 25 \quad &\iff \quad \mathrm{e}^{-(\ln 2)t/30} = 1/16 \\ &\iff \quad -(\ln 2)t/30 = \ln 1/16 = \ln 2^{-4} = -4\ln 2 \\ &\iff \quad t = 4\cdot 30 = \boxed{120 \text{ years}} \end{aligned}$$

Here is another solution, which does not use part (a). By assumptions, the amount of Strontium-90 halves every 30 years. Since $400/25 = 16 = 2^4$, this (halving) needs to happen precisely 4 times, which takes $30\cdot 4 =120$ years

Grading: $400\mathrm{e}^{-(\ln 2)t/30}=25$ is 2pts; 3pts for the simplification; 2pts off if the units are missing or incorrect; correct answer with no explanation is 3pts; penalty for the answer not in a simplified form, e.g. with $\ln(1/16)/\ln(1/2)$. 3pts bonus for the alternative solution with proper justification (which could be worded differently from above); so up to 8 points before any reductions (such as for the missing units).

Problem 3 (15pts)

(a; 8pts) *Find the general solution of the differential equation*

$$y'' + 3y' = 0, \qquad y = y(x).$$

This is the same as $y'' + 3y' + 0y = 0$, so the associated quadratic equation is

$$r^2 + 3r + 0 = 0.$$

This gives $r(r+3) = 0$, so the roots are $r = 0, -3$ and the general solution of the differential equation is

$$y(x) = C_1 \mathrm{e}^{0x} + C_2 \mathrm{e}^{-3x} = \boxed{C_1 + C_2 \mathrm{e}^{-3x}}$$

Grading: 1pt for setting up a reasonable quadratic polynomial; 2pts more if it is correct; 2pts for the correct roots *based on* the polynomial written down (no carryover penalty); 1pt each for $y(x)$ containing the correct basis functions *based on* the roots found; 1pt for the final answer. A loss of at least 3pts if the answer does not contain precisely two constants and another 3pts if the answer contains e^0 (as opposed to being replaced by 1). If no explanation is given, 4pts for the correct answer; 1pt if the answer contains only one of the correct basis elements and nothing else.

(b; 7pts) *Find the solution of the initial-value problem*

$$y'' + 3y' = 0, \quad y(0) = 0, \quad y'(0) = 3, \qquad y = y(x).$$

Starting with the general solution found in (a), we find the constants C_1 and C_2 so that the two initial conditions hold. In order to do this, compute $y'(x)$, $y(0)$, and $y'(0)$ and set them equal to the given initial condition:

$$y'(x) = 0 + C_2 \mathrm{e}^{-3x} \cdot (-3), \quad y(0) = C_1 + C_2 \cdot \mathrm{e}^{-3 \cdot 0} = C_1 + C_2, \quad y'(0) = -3C_2 \cdot \mathrm{e}^{-3 \cdot 0} = -3C_2\,,$$

$$\Longrightarrow \quad \begin{cases} y(0) = C_1 + C_2 = 0 \\ y'(0) = -3C_2 = 3 \end{cases} \quad \Longrightarrow \quad C_2 = -1,\ C_1 = 1.$$

So the solution to the initial-value problem is thus

$$\boxed{y(x) = 1 - \mathrm{e}^{-3x}}$$

Grading: 1pt for computing $y(0)$ and $y'(0)$; 1pt more if correctly; 2pts for setting up a linear system based on the initial conditions *and* $y(0)$ and $y'(0)$ found (no carryover penalty); 2pts for finding C_1 and C_2 based on the linear system set up; final answer based on the solutions of the system found. If part (a) is left blank, then (b) is to be graded out of 15 points if starting without a guess for the general solution.

Problem 4 (20pts)

Let $y = y(x)$ be the solution to the initial-value problem

$$y' = x + 2y, \quad y = y(x), \qquad y(0) = 0.$$

(a; 15pts) *Use Euler's method with $n = 3$ steps to estimate the value of $y(1)$. Show your steps clearly and use simple fractions (so 5/4 or $\frac{5}{4}$, not 1.25).*

The step size is $h = (1-0)/3 = 1/3$, so we need to obtain estimates y_1, y_2, y_3 for the y-value at $x_1 = 1/3$, $x_2 = 2/3$, and $x_3 = 1$ starting with the value $y_0 = 0$ of y at $x_0 = 0$:

$$\begin{aligned}
s_0 &= x_0 + 2y_0 = 0 + 2\cdot 0 = 0 & &\Longrightarrow & y_1 &= y_0 + s_0 h = 0 + 0 = 0\\
s_1 &= x_1 + 2y_1 = \frac{1}{3} + 2\cdot 0 = \frac{1}{3} & &\Longrightarrow & y_2 &= y_1 + s_1 h = 0 + \frac{1}{9} = \frac{1}{9}\\
s_2 &= x_2 + 2y_2 = \frac{2}{3} + 2\cdot\frac{1}{9} = \frac{8}{9} & &\Longrightarrow & y_3 &= y_2 + s_2 h = \frac{1}{9} + \frac{8}{27} = \frac{3+8}{27} = \boxed{\frac{11}{27}}
\end{aligned}$$

Alternatively, this can be done using a table:

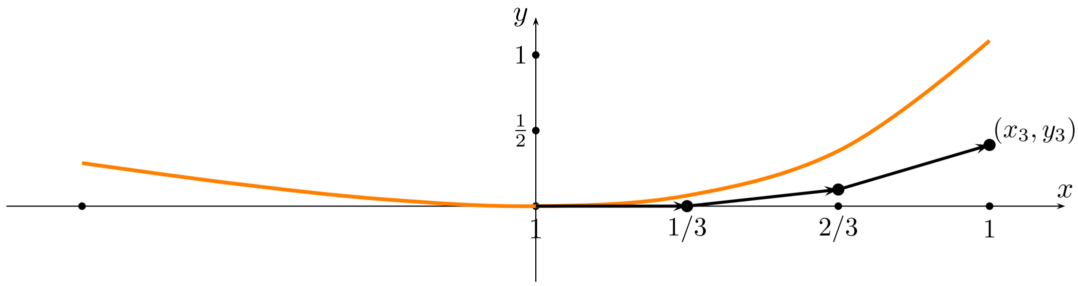
i	x_i	y_i	$s_i = x_i + 2y_i$	$y_{i+1} = y_i + \frac{1}{3}s_i$
0	0	0	$0 + 2\cdot 0 = 0$	$0 + 0 = 0$
1	$\frac{1}{3}$	0	$\frac{1}{3} + 2\cdot 0 = \frac{1}{3}$	$0 + \frac{1}{9} = \frac{1}{9}$
2	$\frac{2}{3}$	$\frac{1}{9}$	$\frac{2}{3} + 2\cdot\frac{1}{9} = \frac{8}{9}$	$\frac{1}{9} + \frac{8}{27} = \frac{3+8}{27} = \boxed{\frac{11}{27}}$

The first column consists of the numbers i running from 0 to $n-1$, where n is the number of steps (3 in this case). The second column starts with the initial value of x (0 in this case) with subsequent entries in the column obtained by adding the step size h ($\frac{1}{3}$ in this case); it ends just before the final value of x would have been entered ($1 = \frac{2}{3} + \frac{1}{3}$ in this case). Thus, the first two columns can be filled in at the start. The first entry in the third column is the initial y-value (0 in the case). After this, one computes the first entries in the remaining two columns and copies the first entry in the last column to the third column in the next line. The process then repeats across the second and third rows. The estimate for the final value of y is the last entry in the table.

Grading: h, x_0, x_1, x_2 1 pt each; correct recursive setup 5pts (1pt each for number of steps, slope and change equations, left end points, end of the procedure); each of 2 steps in each of 3 computations 1pt each

Solution to this problem continues on the next page.

(b; 5pts) *Sketch the path in the xy-plane that represents the approximation carried out in part (a) and indicate its (path's) primary relation to the graph of the actual solution $y=y(x)$ of the initial-value problem above.*



This path consists of the three connected line segments from (x_i, y_i) to (x_{i+1}, y_{i+1}) with $i=0,1,2$. The first of these is tangent to the solution curve at $(0,0)$.

Grading: 3 line segments 1pt, correct end points 2pts, solution curve through (0,0) 1pt, tangent to the 1st line segment 1pt; axes not labeled 1pt off; comments not required

(c; bonus 5pts) *Use Euler's method with $n=3$ steps to estimate the value of $y(-1)$. Show your steps clearly and use simple fractions.*

The step size is still $h=(0-(-1))/3=1/3$, but we now move backwards. Thus, we need to obtain estimates y_1, y_2, y_3 for the y-value at $x_1=-1/3$, $x_2=-2/3$, and $x_3=-1$ starting with the value $y_0=0$ of y at $x_0=0$:

$$\begin{aligned}
s_0 = x_0 + 2y_0 = 0 + 2\cdot 0 = 0 \quad &\Longrightarrow \quad y_1 = y_0 - s_0 h = 0 - 0 = 0\\
s_1 = x_1 + 2y_1 = -\frac{1}{3} + 2\cdot 0 = -\frac{1}{3} \quad &\Longrightarrow \quad y_2 = y_1 - s_1 h = 0 + \frac{1}{9} = \frac{1}{9}\\
s_2 = x_2 + 2y_2 = -\frac{2}{3} + 2\cdot\frac{1}{9} = -\frac{4}{9} \quad &\Longrightarrow \quad y_3 = y_2 - s_2 h = \frac{1}{9} + \frac{4}{27} = \boxed{\frac{7}{27}}
\end{aligned}$$

Alternatively, this can be done using a table:

i	x_i	y_i	$s_i = x_i + 2y_i$	$y_{i+1} = y_i - \frac{1}{3}s_i$
0	0	0	$0 + 2\cdot 0 = 0$	$0 - 0 = 0$
1	$-\frac{1}{3}$	0	$-\frac{1}{3} + 2\cdot 0 = -\frac{1}{3}$	$0 + \frac{1}{9} = \frac{1}{9}$
2	$-\frac{2}{3}$	$\frac{1}{9}$	$-\frac{2}{3} + 2\cdot\frac{1}{9} = -\frac{4}{9}$	$\frac{1}{9} + \frac{4}{27} = \boxed{\frac{7}{27}}$

The first column consists of the numbers i running from 0 to $n-1$, where n is the number of steps (3 in this case). The second column starts with the initial value of x (0 in this case) with subsequent entries in the column obtained by subtracting the step size h ($\frac{1}{3}$ in this case); it ends just before the final value of x would have been entered ($-1=-\frac{2}{3}-\frac{1}{3}$ in this case).

Grading: 0, 4, or 5pts; must be essentially correct, with clear indication of at what points the slopes are taken and how; same with the change equations; 4pts if there is a minor computational misstatement, but completely correct otherwise.

(d; bonus 10pts) *Find the exact value of* $y(-1)$.

First use the *Integrating Factor* method Problem B to find the general solution of the ODE

$$y' - 2y = x.$$

The integrating factor in this case is

$$h(x) = \mathrm{e}^{\int_0^x (-2)\mathrm{d}s} = \mathrm{e}^{-2x}\,.$$

Multiplying both sides of the above equation by $h(x)$, we obtain

$$\big(\mathrm{e}^{-2x}y\big)' = \mathrm{e}^{-2x}y - 2\mathrm{e}^{-2x}y = \mathrm{e}^{-2x}x\,.$$

Using integration by parts, this gives

$$\mathrm{e}^{-2x}y = \int \mathrm{e}^{-2x}x\mathrm{d}x = -\frac{1}{2}\int x\mathrm{d}\mathrm{e}^{-2x} = -\frac{1}{2}\bigg(x\mathrm{e}^{-2x} - \int \mathrm{e}^{-2x}\mathrm{d}x\bigg) = -\frac{1}{2}\bigg(x\mathrm{e}^{-2x} + \frac{1}{2}\mathrm{e}^{-2x} + C\bigg)\,.$$

From $y(0)=0$, we obtain

$$\mathrm{e}^{-2\cdot 0}\cdot 0 = -\frac{1}{2}\bigg(0\cdot\mathrm{e}^{-2\cdot 0} + \frac{1}{2}\mathrm{e}^{-2\cdot 0} + C\bigg).$$

Thus, $C=-\frac{1}{2}$ above and

$$\mathrm{e}^{-2(-1)}y(-1) = -\frac{1}{2}\bigg(-\mathrm{e}^{-2(-1)} + \frac{1}{2}\mathrm{e}^{-2(-1)} - \frac{1}{2}\bigg) = \frac{1}{4}\big(\mathrm{e}^2+1\big) \quad\Longrightarrow\quad \boxed{y(-1) = \frac{1}{4}\big(1+\mathrm{e}^{-2}\big)}$$

Grading: 5pts for applying the Integrating Factor method correctly; 2pts for carrying out the integration-by-parts correctly; 1pt for finding C; 1pt for plugging in $x=-1$; 1pt for the final answer in a simple form; no credit unless the Integrating Factor or Undetermined Coefficients method is used correctly or nearly so.

Problem 5 (20pts)

(a; 3pts) *What are the constant solutions of the differential equation*

$$y' - (y^2-1)x = 0, \qquad y = y(x) \ \ ?$$

The constant solutions correspond to the values of y so that $y'=0$ and thus

$$0 = (y^2-1)x = (y-1)(y+1)x$$

for *all* x. Thus, the constant solutions are $\boxed{y(x) = -1 \text{ and } y(x) = 1}$

Grading: correct answer 2pts (can be written as $y = \pm 1$, etc.); minimal explanation, such as setting $y' = 0$, 1pt; listing $y = -1$ or $y = 1$ only 1pt total, regardless of explanation; listing $x=0$ as constant -5pts from the score received otherwise.

(b; 10pts) *Find the general solution to the differential equation in (a).*

The differential equation becomes separable after moving $(y^2-1)x$ to RHS. After doing this, we move all terms involving y to LHS and all terms involving x to RHS, and integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (y^2-1)x \quad \Longleftrightarrow \quad \frac{\mathrm{d}y}{y^2-1} = x\mathrm{d}x \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}y}{y^2-1} = \int x\mathrm{d}x = \frac{1}{2}x^2 + C.$$

In order to do the y-integral, we need to use (*quick*) **partial fractions** to break the fraction into two with *linear* denominators:

$$\frac{1}{y^2-1} = \frac{1}{(y-1)(y+1)} = \frac{1}{+1-(-1)}\left(\frac{1}{y-1} - \frac{1}{y+1}\right) = \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)$$

This *quick partial fractions* approach gives the answer *immediately* without solving for the coefficients. QPF works in this case because the coefficients of y in the two factors are the same (both are 1 here). It would not have worked if they were different or if *one* of the y's were replaced by something else, such as y^3 (you would have to use the approach of Section 5.7 then). It does not matter in what order the two fractions are written in the third expression, as long as the constant terms are copied to the denominator of the external fraction **without the arrows crossing**. Thus,

$$\int \frac{\mathrm{d}y}{y^2-1} = \frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right)\mathrm{d}y = \frac{1}{2}\big(\ln|y-1| - \ln|y+1|\big) + C' = \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| + C'.$$

Combining this with the RHS integral, we obtain

$$\ln\left|\frac{y-1}{y+1}\right| = x^2 + C \quad \Longleftrightarrow \quad \left|\frac{y-1}{y+1}\right| = \mathrm{e}^{x^2+C} = \mathrm{e}^{x^2}\mathrm{e}^{C} \quad \Longleftrightarrow \quad \frac{y-1}{y+1} = \pm A\mathrm{e}^{x^2} = C\mathrm{e}^{x^2}.$$

This equation defines y implicitly as a function of x, but it can be simplified further. Since

$$\frac{y-1}{y+1} = \frac{y+1-2}{y+1} = 1 - \frac{2}{y+1},$$

the implicit definition of $y = y(x)$ above gives

$$1 - \frac{2}{y+1} = C\mathrm{e}^{x^2} \quad \Longleftrightarrow \quad 1 - C\mathrm{e}^{x^2} = \frac{2}{y+1} \quad \Longleftrightarrow \quad \frac{2}{1-C\mathrm{e}^{x^2}} = y+1$$

$$\Longleftrightarrow \quad y = \frac{2}{1-C\mathrm{e}^{x^2}} - 1 = \frac{1+C\mathrm{e}^{x^2}}{1-C\mathrm{e}^{x^2}}.$$

The constant solution $y=1$ corresponds to $C=0$, but the constant solution $y=-1$ has to be listed separately (in a sense it corresponds to $C=\pm\infty$). So the general solution of the differential equation is

$$\boxed{y(x)=-1, \quad y(x)=\frac{2}{1-Ce^{x^2}}-1=\frac{1+Ce^{x^2}}{1-Ce^{x^2}}}$$

Either version of the family of solutions is fine, including with C replaced by $-C$.

Grading: "integrating" the equation to something like $y=((1/3)y^3-y)x$ or $y=(y^2-1)x^2$ 0pts for the entire question; separation of variables 1pt; RHS integral 1pt; LHS integral 4pts with penalties for computational errors; missing absolute value -1pt; LHS is "integrated" to something like $\ln|y^2-1|$ loss of entire 4pts; simplification from anti-derivatives to the final form 3pts; "simplifying" e^{x^2+C} to $e^{x^2}+e^C$, etc. loss of entire 3pts; incorporating the constant solutions into the final answer 1pt (if they are listed twice or $y=1$ is not absorbed into the formula -2pts); if separation of variables is done incorrectly in a significant way (as opposed to minor copying errors, such as factors of 2) 5pts max for the entire question (with reductions for missing constant solutions, etc.)

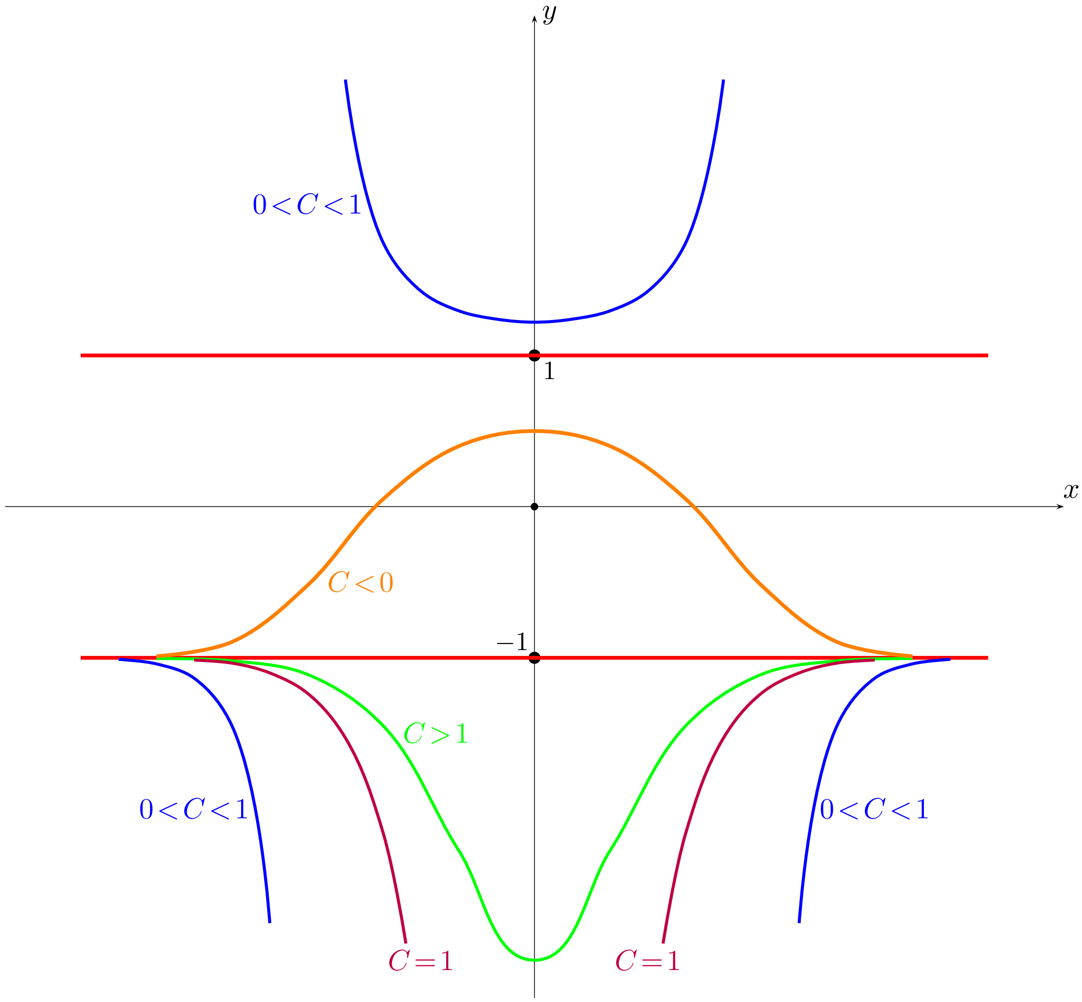
(c; 7pts) *Sketch at least five solution curves, on the same plot of the xy-plane, representing every possible type of behavior of the solutions $y=y(x)$ to the differential equation in (a). Justify the features exhibited on your plot. You can add comments to your plot to more clearly identify the features it is meant to exhibit.*

The two most important solution curves are the horizontal lines $y=\pm1$, corresponding to the constant solutions of the equation. Because of x^2 in the solution in (b), all solution curves are symmetric about the y-axis; this can also be seen directly from the differential equation.

We first look at the behavior of the solution curves in the region $-1<y<1$. Since $y'=(y^2-1)x$, $y'>0$ if $x<0$ and $y'<0$ if $x>0$. Thus, the solution curves in this region ascend to the left of the y-axis and descend to the right of the y-axis, peaking on the y-axis. As $x\longrightarrow\pm\infty$, they approach the line $y=-1$ from above. They correspond to the solutions $y=y(x)$ in (b) with $C<0$ and are defined for all x.

We next consider the solution curves in the region $y>1$. Since $y'=(y^2-1)x$, $y'<0$ if $x<0$ and $y'>0$ if $x>0$. Thus, the solution curves in this region descend to the left of the y-axis and ascend to the right of the y-axis, bottoming out on the y-axis. They rise rapidly to infinity on both sides of the y-axis. The rise happens in a finite time as these curves correspond to the solutions with $0<C<1$, which are not defined for $x=\pm\sqrt{\ln(1/C)}$.

Finally, we consider the solution curves in the region $y<-1$. Since $y'=(y^2-1)x$, $y'<0$ if $x<0$ and $y'>0$ if $x>0$. Thus, the solution curves in this region descend to the left of the y-axis and ascend to the right of the y-axis. As $x\longrightarrow\pm\infty$, they approach the line $y=-1$ from below. Some of these solution curves bottom out on the y-axis and are defined for all x; they correspond to the solutions with $C>1$. Other solution curves in this region drop to $-\infty$ before reaching the y-axis from either the left or the right and are not defined for all x; they correspond to the solutions with $0<C\le1$. The graph of the solution $y=y(x)$ with $C=1$ has two pieces below the line $y=-1$ which drop to $-\infty$ along the y-axis. The graph of a solution with $0<C<1$ has two pieces below the line $y=-1$ which drop to $-\infty$ along the vertical lines $x=\pm\sqrt{\ln(1/C)}$ and a "parabola-like" piece above the line $y=1$ squeezed between the vertical lines $x=-\sqrt{\ln(1/C)}$ and $x=\sqrt{\ln(1/C)}$.



Grading: graphs of constant solutions with y-intercepts labeled 2pts; solution curves in 3 regions with correct behavior 1pt each; explanation up to 2pts; 1pt off if the axes are not labeled; up to 5pts bonus for clear identification and explanation of the vertical asymptotes and of the correspondence between the different pieces in different regions with the values of C in (b)