MAT 127: Calculus C, Fall 2009
Solutions to Midterm I

Problem 1 (20pts)

(a; **8pts**) *Show that the function* $y(x) = 2\mathrm{e}^x$ *is a solution to the initial-value problem*

$$y'' - 6y' + 5y = 0, \quad y = y(x), \qquad y(0) = 2, \qquad y'(0) = 2.$$

Compute $y'(x)$ and $y''(x)$ to check that the differential equation is satisfied:

$$\begin{aligned} y(x) = 2\mathrm{e}^x \quad &\Longrightarrow \quad y'(x) = 2\mathrm{e}^x \quad \Longrightarrow \quad y''(x) = 2\mathrm{e}^x \\ &\Longrightarrow \quad y'' - 6y' + 5y = 2\mathrm{e}^x - 6\cdot 2\mathrm{e}^x + 2\cdot 5\mathrm{e}^x = 0\cdot \mathrm{e}^x = 0. \ \checkmark \end{aligned}$$

It remains to check that the initial condition are satisfied:

$$y(0) = 2\mathrm{e}^0 = 2 \ \checkmark, \qquad y'(0) = 2\mathrm{e}^0 = 2 \ \checkmark.$$

(b; **10pts**) *Find the general solution* $y{=}y(x)$ *to the differential equation*

$$y'' - 6y' + 5y = 0, \qquad y = y(x).$$

The associated polynomial equation is

$$r^2 - 6r + 5 = 0 \qquad \Longleftrightarrow \qquad (r-1)(r-5) = 0.$$

So the two roots are $r_1, r_2 = 1, 5$ and the general solution is $\boxed{y(x) = C_1\mathrm{e}^x + C_2\mathrm{e}^{5x}}$

(c; **2pts**) *What is the relation between the solution given in (a) and the general solution in (b)?*

The solution in (a) is the $C_1 = 2$, $C_2 = 0$ case of the general solution.

Problem 2 (20pts)

A bacteria culture grows at a rate proportional to its size. It contained 125 cells at 8pm and 250 cells at 8:20pm.

(a; **12pts**) *Find an expression for the number of cells in the culture t minutes after 8pm.*

Let $y(t)$ be the number of cells in the culture t minutes after 8pm; so $y(0) = 125$. Since the growth rate is proportional to y, $y(t)$ satisfies the exponential growth equation:

$$y(t) = y(0)\mathrm{e}^{rt} = 125\mathrm{e}^{rt},$$

where r is the relative growth rate. By the last assumption,

$$y(20) = 125\mathrm{e}^{r\cdot 20} = 250 \quad\Longleftrightarrow\quad \mathrm{e}^{r\cdot 20} = 2 \quad\Longleftrightarrow\quad 20r = \ln 2 \quad\Longleftrightarrow\quad r = (\ln 2)/20.$$

Thus, $\boxed{y(t) = 125\mathrm{e}^{(\ln 2)t/20} = 125 \cdot 2^{t/20}}$

Note: In this case, it is not necessary to specify the units for either $y(t)$ or t, but specifying them is fine also. Either of the two expressions for $y(t)$ in the box is fine.

(b; **8pts**) *When will the bacteria culture reach 2,000 cells?*

We need to find t so that

$$\begin{aligned} y(t) = 125\mathrm{e}^{(\ln 2)t/20} = 2000 \quad\Longleftrightarrow\quad \mathrm{e}^{(\ln 2)t/20} = 16 \quad&\Longleftrightarrow\quad (\ln 2)t/20 = \ln 16 = \ln 2^4 = 4\ln 2 \\ &\Longleftrightarrow\quad t = 80. \end{aligned}$$

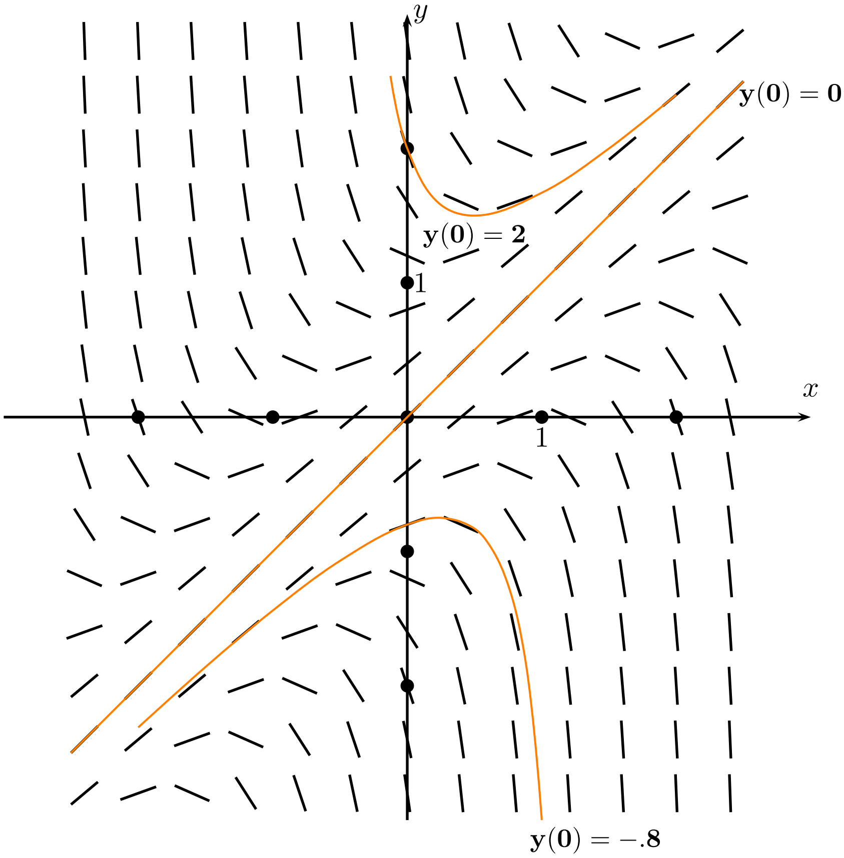
So the bacteria culture reaches 2000 cells 80 minutes after 8pm, which is $\boxed{\text{9:20pm}}$

Here is another solution, which does not use part (a) (**3pts bonus**, whether or not the standard solution is present). By assumptions, the number of cells doubles every 20 minutes. Since $2000/125 = 16 = 2^4$, this (doubling) needs to happen precisely 4 times, which takes $20 \cdot 4 = 80$ minutes. So the bacteria culture reaches 2000 cells 80 minutes after 8pm, which is $\boxed{\text{9:20pm}}$

Note: In this case, the units *pm* or *p.m.*, with or without space after the numbers, are necessary.

Problem 3 (15pts)

The direction field for a differential equation is shown below.



(a; 10pts) *On the direction field, sketch and clearly label the graphs of the three solutions with the initial conditions* $y(0) = -.8$, $y(0) = 0$, *and* $y(0) = 2$ *(each of these three conditions determines a solution to the differential equation).*

The first curve must pass through the point $(0, -.8)$, the second through $(0, 0)$, and the third through (0,2). These curves must be tangent to the slope lines at those points and roughly approximate the slopes otherwise. The sketch suggests that the middle curve is the line $y = x$.

(b; 5pts) *The direction field above is for one of the following differential equations for* $y = y(x)$*:*

$$\text{(i) } y' = 1 - y^2\,, \qquad \text{(ii) } y' = 1 - (x - y)^2\,, \qquad \text{(iii) } y' = 1 - (x + y)^2\,.$$

Which of these three equations does the direction field correspond to and why?

It is not (i), since the direction field for (i) does not depend on x and so the slopes are constant in the horizontal direction. It looks like it might be (ii) because $y(x) = x$ is a solution of (ii). This function is not a solution of (iii). Furthermore, the direction field for (iii) has slopes of 1 along the line $y = -x$; this is clearly not the case in the picture (the slopes are not even constant along this line). So the answer is $\boxed{\text{(ii)}}$

Problem 4 (20pts)

(a; 15pts) *Let* $y = y(x)$ *be the solution to the initial-value problem*

$$y' = xy, \quad y = y(x), \qquad y(0) = 1.$$

Use Euler's method with $n = 3$ *steps to estimate the value of* $y(1)$*. Show your steps clearly and use simple fractions (so 5/4 or* $\frac{5}{4}$*, not 1.25).*

The step size is $h = (1-0)/3 = 1/3$, so we need to obtain estimates y_1, y_2, y_3 for the y-value at $x_1 = 1/3$, $x_2 = 2/3$, and $x_3 = 1$:

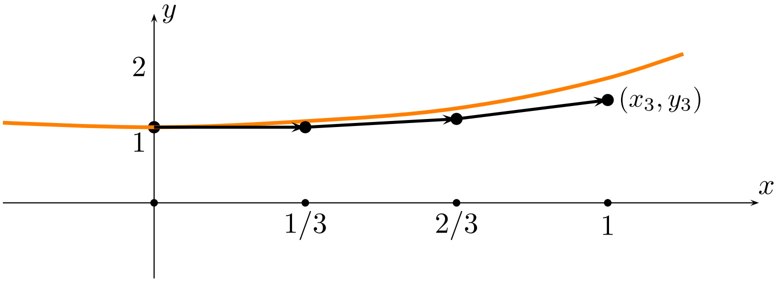
$$\begin{aligned}
s_0 &= x_0 y_0 = 0 && \Longrightarrow && y_1 = y_0 + s_0 h = 1 \\
s_1 &= x_1 y_1 = \frac{1}{3} && \Longrightarrow && y_2 = y_1 + s_1 h = 1 + \frac{1}{9} = \frac{10}{9} \\
s_2 &= x_2 y_2 = \frac{20}{27} && \Longrightarrow && y_3 = y_2 + s_2 h = \frac{10}{9} + \frac{20}{81} = \frac{90+20}{81} = \boxed{\frac{110}{81}}
\end{aligned}$$

Alternatively, this can be done using a table:

i	x_i	y_i	$s_i = x_i y_i$	$y_{i+1} = y_i + \frac{1}{3} s_i$
0	0	1	0	$1 + 0 = 1$
1	$\frac{1}{3}$	1	$\frac{1}{3}$	$1 + \frac{1}{9} = \frac{10}{9}$
2	$\frac{2}{3}$	$\frac{10}{9}$	$\frac{20}{27}$	$\frac{10}{9} + \frac{20}{81} = \frac{90+20}{81} = \boxed{\frac{110}{81}}$

The first column consists of the numbers i running from 0 to $n-1$, where n is the number of steps (3 in this case). The second column starts with the initial value of x (0 in this case) with subsequent entries in the column obtained by adding the step size h ($\frac{1}{3}$ in this case); it ends just before the final value of x would have been entered ($1 = \frac{2}{3} + \frac{1}{3}$ in this case). Thus, the first two columns can be filled in at the start. The first entry in the third column is the initial y-value (1 in the case). After this, one computes the first entries in the remaining two columns and copies the first entry in the last column to the third column in the next line. The process then repeats across the second and third rows. The estimate for the final value of y is the last entry in the table.

(b; 5pts) *Sketch the path in the xy-plane that represents the approximation carried out in part (a) and indicate its (path's) primary relation to the graph of the actual solution* $y = y(x)$ *of the initial-value problem in (a).*



This path consists of the three connected line segments from (x_i, y_i) to (x_{i+1}, y_{i+1}) with $i = 0, 1, 2$. The first of these is tangent to the solution curve at $(0, 1)$.

3pt bonus: The actual solution curve always lies above the path. This is because

$$y'' = y + xy' = y + x^2y = (1 + x^2)y;$$

so the second derivative is positive in the first quadrant.
All or nothing bonus; explanation is required.

Problem 5 (25pts)

Find the general solution $y = y(x)$ to the differential equation

$$y' + 9 = y^2, \qquad y = y(x).$$

Sketch at least five solution curves, on the same plot of the xy-plane, representing every possible type of behavior of the solutions $y = y(x)$ depending on the value of $y(0)$; justify the features exhibited on your plot (this can be done with or without using the general solution). You can add comments to your plot to more clearly identify the features it is meant to exhibit.

The differential equation becomes separable after moving the 9 to RHS. After doing this, we move all terms involving y to LHS and all terms involving x to RHS, and integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 9 \quad \Longleftrightarrow \quad \frac{\mathrm{d}y}{y^2 - 9} = \mathrm{d}x \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}y}{y^2 - 9} = \int \mathrm{d}x = x + C.$$

In order to do the y-integral, we need to use partial fractions:

$$\frac{1}{y^2 - 9} = \frac{1}{(y-3)(y+3)} = \frac{1}{3-(-3)}\left(\frac{1}{y-3} - \frac{1}{y+3}\right) = \frac{1}{6}\left(\frac{1}{y-3} - \frac{1}{y+3}\right).$$

Thus,

$$\int \frac{\mathrm{d}y}{y^2 - 9} = \frac{1}{6}\int \left(\frac{1}{y-3} - \frac{1}{y+3}\right)\mathrm{d}y = \frac{1}{6}\big(\ln|y-3| - \ln|y+3|\big) + C' = \frac{1}{6}\ln\left|\frac{y-3}{y+3}\right| + C'.$$

Combining this with the previous line gives

$$\ln\left|\frac{y-3}{y+3}\right| = 6x + C \quad \Longleftrightarrow \quad \left|\frac{y-3}{y+3}\right| = \mathrm{e}^{6x+C} = \mathrm{e}^{6x}\mathrm{e}^{C} \quad \Longleftrightarrow \quad \frac{y-3}{y+3} = \pm A\mathrm{e}^{6x} = C\mathrm{e}^{6x}.$$

This equation defines y implicitly as a function of x, but it can be simplified further. Since

$$\frac{y-3}{y+3} = \frac{y+3-6}{y+3} = 1 - \frac{6}{y+3},$$

the implicit definition of $y = y(x)$ above gives

$$1 - \frac{6}{y+3} = C\mathrm{e}^{6x} \quad \Longleftrightarrow \quad 1 - C\mathrm{e}^{6x} = \frac{6}{y+3} \quad \Longleftrightarrow \quad \frac{6}{1 - C\mathrm{e}^{6x}} = y + 3$$

$$\Longleftrightarrow \quad y = \frac{6}{1 - C\mathrm{e}^{6x}} - 3 = 3\frac{1 + C\mathrm{e}^{6x}}{1 - C\mathrm{e}^{6x}}.$$

We now have to go back to the equation and check for the constant solutions. These are the solutions of $y' = y^2 - 9 = 0$, so $y = \pm 3$. The $y = 3$ solution is covered by the $C = 0$ case above; the other one has to be listed separately (in a sense it corresponds to $C = \pm\infty$). So the general solution of the differential equation is

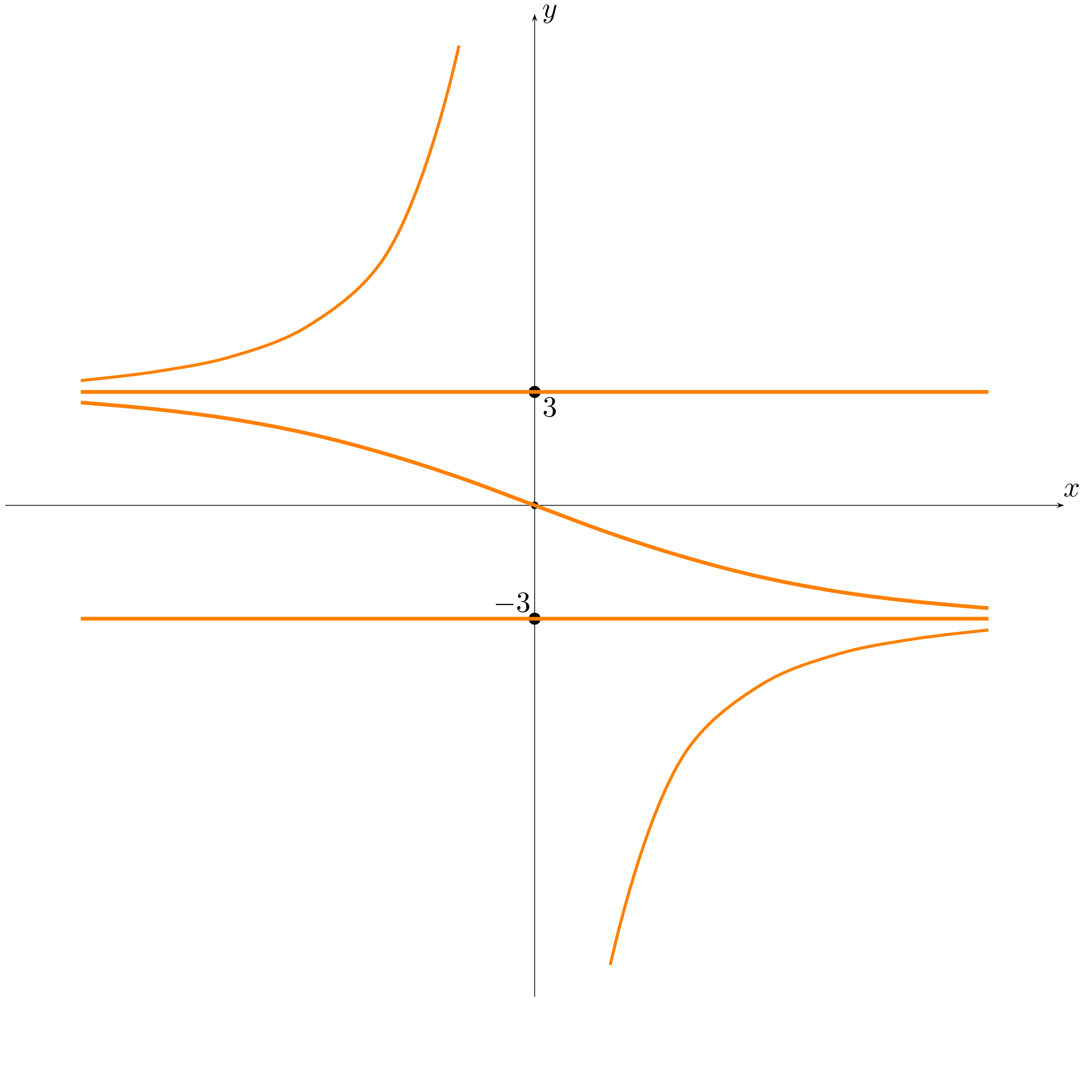
$$\boxed{y(x) = -3, \quad y(x) = \frac{6}{1 - C\mathrm{e}^{6x}} - 3 = 3\frac{1 + C\mathrm{e}^{6x}}{1 - C\mathrm{e}^{6x}}}$$

Either version of the family of solutions is fine, including with C replaced by $-C$.

The two most important solution curves are the horizontal lines $y = \pm 3$, corresponding to the constant solutions of the equation. These solutions pretty much must be obtained directly from the differential equation.

Since $y' = y^2 - 9 = (y-3)(y+3)$, $y' > 0$ if $y < -3$ or $y > 3$. Thus, in the regions $y < -3$ and $y > 3$, the solution curves rise. In the region $y < -3$, the slopes increase as y becomes more negative and flatten out as y approaches -3 (from below). Thus, the solution curves asymptotically approach the horizontal line $y = -3$ as $y \longrightarrow \infty$ and descend rapidly as y decreases. Similarly, in the region $y > 3$, the slopes increase as y increases and flatten out as y approaches 3 (from above). Thus, the solution curves asymptotically approach the horizontal line $y = 3$ as $y \longrightarrow -\infty$ and ascend rapidly as y increases.

Since $y' = y^2 - 9 = (y-3)(y+3)$, $y' < 0$ if $-3 < y < 3$. Thus, in the region $-3 < y < 3$, the solution curves descend. Their slopes flatten out as y approaches -3 (from above) and 3 (from below). Thus, the solution curves asymptotically approach the horizontal line $y = -3$ as $x \longrightarrow \infty$ and the horizontal line $y = 3$ as $x \longrightarrow -\infty$.



The above can also be seen from the explicit analytic solution

$$y(x) = \frac{6}{1 - Ce^{6x}} - 3.$$

If $C < 0$, $1 - Ce^{6x} > 1$ and so $-3 < y(x) < 3$. In this case, $1 - Ce^{6x}$ increases from 1 to ∞ as x increases from $-\infty$ to ∞. Thus, $y(x) = 6/(1 - Ce^{6x}) - 3$ decreases (being the reciprocal) from $6/1 - 3 = 3$ to $6/\infty - 3 = -3$ as x increases from $-\infty$ to ∞.

If $C > 0$, $1 - Ce^{6x} < 1$. The function $1 - Ce^{6x}$ decreases from 1 to $-\infty$ as x runs from $-\infty$ to ∞ and passes 0 when

$$Ce^{6x} = 1 \qquad \Longleftrightarrow \qquad \ln C + 6x = 0 \qquad \Longleftrightarrow \qquad x = -(\ln C)/6.$$

Thus, $y(x) = \frac{6}{1-Ce^{6x}} - 3$ is not defined for $x = -(\ln C)/6$. If $x < -(\ln C)/6$, $0 < 1 - Ce^{6x} < 1$ and thus $3 < y(x) < \infty$ and $y(x)$ is increasing (because $1 - Ce^{6x}$ is decreasing); $y(x)$ approaches $6/1 - 3 = 3$ as $x \longrightarrow -\infty$. If $x > -(\ln C)/6$, $1 - Ce^{6x} < 0$ and thus $y(x) < -3$ and $y(x)$ is again increasing; $y(x)$ approaches $6/\infty - 3 = -3$ as $x \longrightarrow \infty$. It is sufficient to do this analysis for a specific C (such as $C = 1$ in this paragraph and $C = -1$ in the previous paragraph).

2pt bonus: Since the equation is autonomous, horizontal translates of solution curves are again solution curves. In other words, if $y = y(x)$ is a solution of the differential equation, so is $\tilde{y}(x) = y(x - a)$. This can be seen directly from the differential equation:

$$\tilde{y}'(x) = y'(x - a) = y(x - a)^2 - 9 = \tilde{y}(x)^2 - 9.\checkmark$$

This can also be seen from the family of solutions:

$$y(x) = \frac{6}{1 - Ce^{6x}} - 3 \qquad \Longrightarrow \qquad \tilde{y}(x) = y(x - a) = \frac{6}{1 - Ce^{6(x-a)}} - 3 = \frac{6}{1 - Ce^{-6a}e^{6x}} - 3;$$

so $\tilde{y}$ corresponds to the constant $\tilde{C} = e^{-6a}C$.
All or nothing bonus; first sentence suffices for explanation, as is either of the more detailed explanations; the sketch must indicate some horizontal translates.

3pt bonus: As x increases, solutions $y = y(x)$ with $y(x) > 3$ approach infinity in finite time; as x decreases, solutions $y = y(x)$ with $y(x) < -3$ approach infinity in finite time. This can be seen directly from the differential equation as follows. If $|y|$ is very large, the 9 in differential equation is irrelevant, so the solutions are similar to the solutions of $y' = y^2$; these are $y(x) = 1/(A - x)$ and clearly blow up in a finite time. This can also be seen from the family solution

$$y(x) = \frac{6}{1 - Ce^{6x}} - 3$$

with $C > 0$. As x increases from $-\infty$ to $-(\ln C)/6$, $1 - Ce^{6x}$ decreases from 1 to 0 and so $y(x)$ increases from 3 to ∞; as x decreases from ∞ to $-(\ln C)/6$, $1 - Ce^{6x}$ increases from $-\infty$ to 0 and so $y(x)$ decreases from -3 to $-\infty$.
All or nothing bonus; an explanation is required, but either of the above two is sufficient.

4pt bonus: The configuration of solution curves is symmetric about the origin. In other words, if $y = y(x)$ is a solution of the differential equation, so is $\tilde{y}(x) = -y(-x)$. This can be seen directly from the differential equation:

$$\tilde{y}'(x) = -y'(-x)\cdot(-1) = y(-x)^2 - 9 = \tilde{y}(x)^2 - 9.\checkmark$$

This can also be seen from the family of solutions:

$$y(x) = 3\frac{1+Ce^{6x}}{1-Ce^{6x}} \qquad \Longrightarrow$$
$$\tilde{y}(x) = -y(-x) = -3\frac{1+Ce^{-6x}}{1-Ce^{-6x}} = -3\frac{(1/C)e^{6x}+1}{(1/C)e^{6x}-1} = 3\frac{1+(1/C)e^{6x}}{1-(1/C)e^{6x}};$$

so $\tilde{y}$ corresponds to the constant $\tilde{C} = 1/C$.
All or nothing bonus; an explanation is required, but either of the above two is sufficient.

Grading: If your analytic solution or the sketch was correct and complete, you should have received at least 15 points on this problem. However, the maximum you could have received for both is 25 points (before the bonus points), from which points were subtracted for any errors. The penalty for missing one or both constant solutions is 3 points in either part, with 5 points taken off if they are missed in both parts.