MATHEMATICAL INDUCTION

1. Introduction

Mathematics distinguishes itself from the other sciences in that it is built upon a set of axioms and definitions, on which all subsequent theorems rely. All theorems can be derived, or proved, using the axioms and definitions, or using previously established theorems. By contrast, the theories in most other sciences, such as the Newtonian laws of motion in physics, are often built upon experimental evidence and can never be proved to be true.

It is therefore insufficient to argue that a mathematical statement is true simply by experiments and observations. For instance, Fermat (1601–1665) conjectured that when n is an integer greater than 2, the equation $x^n + y^n = z^n$ admits no solutions in positive integers. Many attempts by mathematicians in finding a counter-example (i.e. a set of positive integer solution) ended up in failure. Despite that, we cannot conclude that Fermat's conjecture was true without a rigorous proof. In fact, it took mathematicians more than three centuries to find the proof, which was finally completed by the English mathematician Andrew Wiles in 1994.

To conclude or even to conjecture that a statement is true merely by experimental evidence can be dangerous. For instance, one might conjecture that $n^2 - n + 41$ is prime for all natural numbers n. One can easily verify this: when $n = 1$, $n^2 - n + 41 = 41$ is prime; when $n = 2$, $n^2 - n + 41 = 43$ is prime, and so on. Even if one continues the experiment until $n = 10$, or even $n = 20$, one would not be able to find a counter-example. However, it is easy to see that the statement is wrong, for when $n = 41$ the expression is equal to 41^2 which definitely is not prime.

While experimental evidence is insufficient to guarantee the truthfulness of a statement, it is often not possible to verify the statement for all possible cases either. For instance, one might conjecture that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all natural numbers n. Of course one easily verifies that the statement is true for the first few (even the first few hundreds or even thousands of cases if one bothers to do so) values of n. Yet we cannot conclude that the statement is true. Maybe it will fail at some unattempted values, who knows? It is not possible to verify the statement for all possible values of n since there are infinitely many of them. So how can we verify the statement? A powerful tool is mathematical induction.

2. The Basic Principle

An analogy of the principle of mathematical induction is the game of dominoes. Suppose the dominoes are lined up properly, so that when one falls, the successive one will also fall. Now by pushing the first domino, the second will fall; when the second falls, the third will fall; and so on. We can see that all dominoes will ultimately fall.

So the basic principle of mathematical induction is as follows. To prove that a statement holds for all positive integers n, we first verify that it holds for $n = 1$, and then we prove that if it holds for a certain natural number k, it also holds for $k+1$. This is given in the following.

Theorem 2.1. (Principle of Mathematical Induction)

Let $S(n)$ denote a statement involving a variable n. Suppose

(1) $S(1)$ is true;

(2)   if $S(k)$ is true for some positive integer k, then $S(k+1)$ is also true.

Then $S(n)$ is true for all positive integers n.

Example 2.1.

Prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all natural numbers n.

Solution.

We shall prove the statement using mathematical induction.

Clearly, the statement holds when $n = 1$ since $1 = 1^2$.

Suppose the statement holds for some positive integer k. That is, $1 + 3 + 5 + \cdots + (2k-1) = k^2$.

Consider the case $n = k+1$.

By the above assumption (which we shall call the **induction hypothesis**), we have

$$1 + 3 + 5 + \cdots + [2(k+1)-1] = [1 + 3 + 5 + \cdots + (2k-1)] + (2k+1)$$
$$= k^2 + (2k+1)$$
$$= (k+1)^2$$

That is, the statement holds for $n = k+1$ provided that it holds for $n = k$.

By the principle of mathematical induction, we conclude that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all natural numbers n.

The principle of mathematical induction can be used to prove a wide range of statements involving variables that take discrete values. Some typical examples are shown below.

Example 2.2.

Prove that $23^n - 1$ is divisible by 11 for all positive integers n.

Solution.

Clearly, $23^1 - 1 = 22$ is divisible by 11.

Suppose $11 \mid 23^k - 1$ for some positive integer k.

Then $23^{k+1} - 1 = 23 \cdot 23^k - 1 = 11 \cdot 2 \cdot 23^k + (23^k - 1)$ which is also divisible by 11.

It follows that $23^n - 1$ is divisible by 11 for all positive integers n.

Example 2.3.

Let $x > -1$ be a real number. Prove that $(1+x)^n \geq 1 + nx$ for all natural numbers n.

Solution.

Clearly, the inequality holds for $n = 1$ since $(1+x)^1 = 1 + 1(x)$.

Suppose $(1+x)^k \geq 1 + kx$ for some positive integer k.

For the case $n = k + 1$, we have

$$
\begin{aligned}
(1+x)^{k+1} &= (1+x)^k (1+x) \\
&\geq (1+kx)(1+x) \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x
\end{aligned}
$$

Hence, if the inequality holds for the case $n = k$, it also holds for the case $n = k + 1$.

It follows that $(1+x)^n \geq 1 + nx$ for all natural numbers n.

(Question: Where in the proof did we make use of the fact that $x > -1$?)

While we have illustrated how mathematical induction can be used to prove certain statements, it should be remarked that many of these statements can actually be proved without using mathematical induction. There are always ingenious ways to prove those statements. The use of mathematical induction, however, provides an easy and mechanical (though sometimes tedious) way of proving a wide range of statements.

3. Variations of the Basic Principle

There are many variations to the principle of mathematical induction. The ultimate principle is the same, as we have illustrated with the example of dominoes, but these variations allow us to prove a much wider range of statements.

Theorem 3.1. (Principle of Mathematical Induction, Variation 1)

Let $S(n)$ denote a statement involving a variable n. Suppose

(1) $S(k_0)$ is true for some positive integer k_0;

(2) if $S(k)$ is true for some positive integer $k \geq k_0$, then $S(k+1)$ is also true.

Then $S(n)$ is true for all positive integers $n \geq k_0$.

In some cases a statement involving a variable n holds when n is 'large enough', but does not hold when, say, $n = 1$. In this case Theorem 2.1 does not apply, but the above variation allows us to prove the statement.

Example 3.1.

Prove that $2^n > n^2$ for all natural numbers $n \geq 5$.

Solution.

First, we check that $2^5 = 32 > 25 = 5^2$, so the inequality holds for $n = 5$.

Suppose $2^k > k^2$ for some integer $k \geq 5$.

Then $2^{k+1} = 2 \cdot 2^k$

$\qquad > 2k^2$

$\qquad > (k+1)^2$

The last inequality holds since $2k^2 - (k+1)^2 = (k-1)^2 - 2 > 0$ whenever $k \geq 5$.

Hence, if the inequality holds for $n = k$, it also holds for $n = k+1$.

By Theorem 3.1, $2^n > n^2$ for all natural numbers $n \geq 5$.

Sometimes a sequence may be defined recursively, and a term may depend on some previous terms. In particular, it may depend on more than one previous terms. In this case Theorem 2.1 does not apply because assuming $S(k)$ holds for a single k is not sufficient. We need the following.

Theorem 3.2. (Principle of Mathematical Induction, Variation 2)

Let $S(n)$ denote a statement involving a variable n. Suppose

(1) $S(1)$ and $S(2)$ are true;

(2)   if $S(k)$ and $S(k+1)$ are true for some positive integer k, then $S(k+2)$ is also true.

Then $S(n)$ is true for all positive integers n.

Of course there is no need to restrict ourselves only to 'two levels'. Moreover, in the spirit of Theorem 3.1, there is no need to start from $n=1$. We leave the formulation as an exercise.

Example 3.2.

Let $\{a_n\}$ be a sequence of natural numbers such that $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all natural numbers n. Prove that $a_n = 2^n + 3^n$ for all natural numbers n.

Solution.

We first check that $a_1 = 5 = 2^1 + 3^1$ and $a_2 = 13 = 2^2 + 3^2$.

Suppose $a_k = 2^k + 3^k$ and $a_{k+1} = 2^{k+1} + 3^{k+1}$ for some natural number k.

Then $a_{k+2} = 5a_{k+1} - 6a_k$
$$= 5\left(2^{k+1} + 3^{k+1}\right) - 6\left(2^k + 3^k\right)$$
$$= 4 \cdot 2^k + 9 \cdot 3^k$$
$$= 2^{k+2} + 3^{k+2}$$

Hence, if the formula holds for $n = k$ and $n = k+1$, it also holds for $n = k+2$.

By Theorem 3.2, $a_n = 2^n + 3^n$ for all natural numbers n.

Sometimes to prove a statement we need to consider the odd cases and even cases separately. To combine them nicely into one single case, we need the following.

Theorem 3.3. (Principle of Mathematical Induction, Variation 3)

Let $S(n)$ denote a statement involving a variable n. Suppose

(1) $S(1)$ and $S(2)$ are true;

(2)   if $S(k)$ is true for some positive integer k, then $S(k+2)$ is also true.

Then $S(n)$ is true for all positive integers n.

Although Theorem 3.2 and Theorem 3.3 look similar, their nature is quite different.

Again there is no need to restrict ourselves to considering only two initial cases, but we do not bother to go into the details.

Example 3.3.

Prove that for all natural numbers n, there exist distinct integers x, y, z for which $x^2 + y^2 + z^2 = 14^n$.

Solution.

For $n = 1$ and $n = 2$, such integers exist as $1^2 + 2^2 + 3^2 = 14$ and $4^2 + 6^2 + 12^2 = 14^2$.

Suppose for $n = k$ (where k is some positive integer), such integers exist, i.e. $x_0^2 + y_0^2 + z_0^2 = 14^k$ for some distinct integers x_0, y_0 and z_0.

Then for $n = k + 2$, such integers also exist because $(14x_0)^2 + (14y_0)^2 + (14z_0)^2 = 14^{k+2}$.

By Theorem 3.3, the result follows.

In Theorem 3.2, we remarked that sometimes assumption of $S(k)$ for a single k may not be sufficient, so we may need to assume the statement holds for two values (and accordingly we need to verify two initial cases). We also remarked that there is no need to restrict ourselves to only two values; we could generalize to any finite number of cases. The following variation gives a further generalization of this, assuming all cases from 1 to k.

Theorem 3.4. (Principle of Mathematical Induction, Variation 4)

Let $S(n)$ denote a statement involving a variable n. Suppose

(1) $S(1)$ is true;

(2)   if for some positive integer k, $S(1)$, $S(2)$, …, $S(k)$ are all true, then $S(k+1)$ is also true.

Then $S(n)$ is true for all positive integers n.

Example 3.4.

(APMO 1999) Let a_1, a_2, … be a sequence of real numbers satisfying $a_{i+j} \le a_i + a_j$ for all $i, j = 1$, 2, … Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \ge a_n$$

for each positive integer n.

Solution.

Clearly, the inequality holds for $n = 1$.

Suppose the inequality holds for $n = 1, 2, \ldots, k$ for some positive integer k.

Then, by adding the inequalities

$$a_1 \geq a_1$$

$$a_1 + \frac{a_2}{2} \geq a_2$$

$$\vdots$$

$$a_1 + \frac{a_2}{2} + \cdots + \frac{a_k}{k} \geq a_k$$

we get $ka_1 + (k-1)\frac{a_2}{2} + \cdots + \frac{a_k}{k} \geq a_1 + a_2 + \cdots + a$,

i.e. $(k+1)\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_k}{k}\right) \geq 2(a_1 + a_2 + \cdots + a_k) = (a_1 + a_k) + (a_2 + a_{k-1}) + \cdots + (a_k + a_1) \geq ka_{k+1}$.

It follows that $(k+1)\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_k}{k} + \frac{a_{k+1}}{k+1}\right) \geq (k+1)a_{k+1}$.

Hence $a_1 + \frac{a_2}{2} + \cdots + \frac{a_{k+1}}{k+1} \geq a_{k+1}$, i.e. the inequality also holds for $n = k+1$.

By Theorem 3.4, the result follows.

Finally, we introduce a special variation, commonly known as **backward induction**.

Theorem 3.5. (Backward Induction)

Let $S(n)$ denote a statement involving a variable n. Suppose

(1) $S(n)$ is true for infinitely many natural numbers n;

(2) if $S(k)$ is true for some positive integer $k > 1$, then $S(k-1)$ is also true.

Then $S(n)$ is true for all positive integers n.

The most typical example where backward induction is used is perhaps in the proof of the **AM-GM inequality**, as shown in the example below.

Example 3.5. (AM-GM inequality)

Prove that for positive integers a_1, a_2, ..., a_n,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} .$$

In other words, the **arithmetic mean** (AM) is always greater than or equal to the **geometric mean** (GM).

Solution.

From $\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geq 0$, we obtain $\dfrac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$, i.e. the inequality holds for $n = 2$.

Suppose the inequality holds when $n = k$ for some positive integer k.

Consider the case $n = 2k$. Using the case $n = 2$ and the induction hypothesis, we have

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_{2k}}{2k} &= \frac{1}{k}\left(\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2} + \cdots + \frac{a_{2k-1} + a_{2k}}{2}\right) \\
&\geq \frac{\sqrt{a_1 a_2} + \sqrt{a_3 a_4} + \cdots + \sqrt{a_{2k-1} + a_{2k}}}{k} \\
&\geq \sqrt[k]{\sqrt{a_1 a_2} \cdot \sqrt{a_3 a_4} \cdots \cdots \sqrt{a_{2k-1} a_{2k}}} \\
&= \sqrt[2k]{a_1 a_2 \cdots a_{2k}}
\end{aligned}$$

i.e. the inequality also holds for $n = 2k$.

By Theorem 2.1, the inequality holds for all positive powers of 2.

In other words, condition (1) in Theorem 3.5 is satisfied.

Again, we suppose the inequality holds when $n = k$ for some positive integer k, i.e.

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \cdots a_k}$$

Applying the substitution $a_k = \dfrac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}$ and simplifying (the details of which are left as an exercise), we get

$$\frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1} \geq \sqrt[k-1]{a_1 a_2 \cdots a_{k-1}}$$

i.e. the inequality also holds when $n = k - 1$.

By Theorem 3.5, the inequality is proved.

4. Miscellaneous Examples

Most of the examples we have seen deal with algebraic (in)equalities and problems in number theory. One should not be misled to think that these are the only areas in which the method of mathematical induction applies. In fact, the method is so powerful that it is useful in almost every branch of mathematics. In this section we shall see some miscellaneous examples.

Example 4.1.

Prove that

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\csc\frac{\theta}{2}$$

for all positive integers n.

Solution.

When $n = 1$, the right hand side is

$$\sin\theta\sin\frac{\theta}{2}\csc\frac{\theta}{2} = \sin\theta.$$

So the formula holds for $n = 1$.

Suppose the formula holds for $n = k$, i.e.

$$\sin\theta + \sin 2\theta + \cdots + \sin k\theta = \sin\frac{(k+1)\theta}{2}\sin\frac{k\theta}{2}\csc\frac{\theta}{2}.$$

Consider the case $n = k+1$. By the induction hypothesis,

$$\sin\theta + \sin 2\theta + \cdots + \sin k\theta + \sin(k+1)\theta$$

$$= \sin\frac{(k+1)\theta}{2}\sin\frac{k\theta}{2}\csc\frac{\theta}{2} + \sin(k+1)\theta$$

$$= \sin\frac{(k+1)\theta}{2}\sin\frac{k\theta}{2}\csc\frac{\theta}{2} + 2\sin\frac{(k+1)\theta}{2}\cos\frac{(k+1)\theta}{2}$$

$$= \sin\frac{(k+1)\theta}{2}\csc\frac{\theta}{2}\left[\sin\frac{k\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{(k+1)\theta}{2}\right]$$

$$= \sin\frac{(k+1)\theta}{2}\csc\frac{\theta}{2}\left[\sin\frac{k\theta}{2} + \sin\left(\frac{\theta}{2} + \frac{(k+1)\theta}{2}\right) + \sin\left(\frac{\theta}{2} - \frac{(k+1)\theta}{2}\right)\right]$$

$$= \sin\frac{(k+1)\theta}{2}\csc\frac{\theta}{2}\left[\sin\frac{k\theta}{2} + \sin\frac{(k+2)\theta}{2} - \sin\frac{k\theta}{2}\right]$$

$$= \sin\frac{[(k+1)+1]\theta}{2}\sin\frac{(k+1)\theta}{2}\csc\frac{\theta}{2}$$

By the principle of mathematical induction, the formula holds for all positive integers n.

Example 4.2.

Prove that $\left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$ is an even integer for all natural numbers n.

Solution.

Write $f(n) = \alpha^n + \beta^n$ where $\alpha = 3+\sqrt{5}$ and $\beta = 3-\sqrt{5}$.

It is straightforward to check that $f(1) = 6$ and $f(2) = 28$ are even integers.

Suppose $f(k)$ and $f(k+1)$ are both even integers for some positive integer k.

Consider the case $n = k+2$.

Note that α and β are roots of the equation $x^2 - 6x + 4 = 0$.

So $\alpha^2 = 6\alpha - 4$ and $\beta^2 = 6\beta - 4$, and thus

$$
\begin{aligned}
f(k+2) &= \alpha^{k+2} + \beta^{k+2} \\
&= \alpha^k (6\alpha - 4) + \beta^k (6\beta - 4) \\
&= 6(\alpha^{k+1} + \beta^{k+1}) - 4(\alpha^k + \beta^k) \\
&= 6f(k+1) - 4f(k)
\end{aligned}
$$

It follows that $f(k+2)$ must also be an even integer.

By mathematical induction, we conclude that $f(n)$ is an even integer for all natural numbers n.

Example 4.3.

Let $n > 1$ be an integer. In a football league there are n teams. Every two teams have played against each other exactly once, and in each match no draw is allowed. Prove that it is possible to number the teams 1, 2, …, n in a way such that team i beats team $i+1$ for $i = 1, 2, …, n-1$.

Solution.

When there are only two teams we simply number the winning team as 1 and the other as 2.

Suppose such a numbering is possible in the case of k teams.

Consider the case with $k+1$ teams.

Take any k teams, and number them 1 to k according to the requirement. This is possible by the induction hypothesis.

Now we try to number the $(k+1)$st team. If it has not won any match, then simply number it as $k+1$. This is possible since it was beaten by the team numbered k.

Suppose it has won at least one match, and team j is the team with the smallest number amongst all the teams that it has beaten. Then we assign the number j to this team, and renumber the original

teams j to n by increasing the number by 1. It is easy to see that this numbering satisfies the requirement.

So if the case with k teams works, the case with $k+1$ teams also works.

By the principle of mathematical induction, the original statement is proved.

Example 4.4.

Prove that, given two or more squares, one can always cut them (using only compasses, straight edge and scissors) and reform them into a large square.

Solution.

In the case of two squares, we resort to the following diagram:
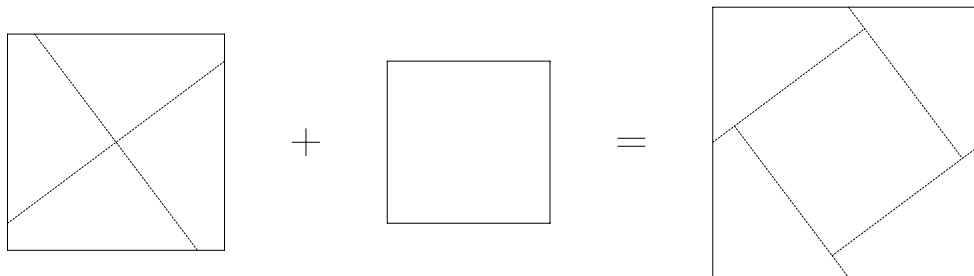
We leave it to the reader to work out how the dotted lines are to be drawn and to verify that such constructions are indeed possible using compasses and straight edge.

Suppose the statement is true in the case of k squares. Then, in the case of $k+1$ squares, we can cut k of the squares to form a large square, according to the induction hypothesis. This leaves only two squares, but we have shown that two squares can be cut to form one large square.

By the principle of mathematical induction, the statement is proved.

Example 4.5.

In a party there are $2n$ participants, where n is a natural number. Some participants shake hands with other participants. It is known that there do not exist three participants who have shaken hands with each other. Prove that the total number of handshakes is not more than n^2.

Solution.

When $n=1$, the number of handshakes is at most $1=1^2$.

Suppose that with $2k$ people, the total number of handshakes is at most k^2 under the given condition.

Consider the case $n=k+1$, i.e. $2k+2$ people. Pick two people who have shaken hands with each other (if no such people exist, then the total number of handshake would be zero), say A and B.

Under the induction hypothesis, there are at most k^2 handshakes among the other $2k$ people. Now by the given condition, none of these $2k$ people have shaken hands with both A and B. So these $2k$ people have at most $2k$ handshakes with A and B. Taking the handshake between A and B into account, the total number of handshakes is at most $k^2 + 2k + 1 = (k+1)^2$.

By the principle of mathematical induction, the result follows.

Example 4.6.

There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around the track in the clockwise direction.

Solution.

The case $n = 1$ is trivial.

Suppose the statement holds when $n = k$.

Consider the case $n = k + 1$.

First, observe that there is a car A which can reach the next car B in the clockwise direction, for otherwise the gas for all cars will not be able to complete a lap.

Now we empty the gas of B into A and remove A. There are k cars left. By the induction hypothesis, there is a car which can complete a lap. Putting back car A, this car will also be able to complete the lap because when it gets to car A, the gas collected will be enough to get it to car B.

By mathematical induction, the statement is true for all positive integers n.

5. Higher Dimensional Induction

So far we have been dealing with statements involving a single variable n. In mathematics, statements often involve more than one variables. With some modification, the principle of mathematical induction can still be applied to prove certain such statements. In this section we will see examples of **two-dimensional induction**. Higher-dimensional induction can be dealt with similarly.

We motivated the principle of (one-dimensional) mathematical induction using the example of dominoes. The traditional dominoes are lined up in a straight line, analogous to one-dimensional induction. In recent years, however, more and more variations have been introduced, some having

the dominoes with different colours arranged to form a nice picture. With this more complicated set-up, we need to determine more carefully how to make sure that all dominoes fall under certain conditions. In the same way we need to formulate a rule for two-dimensional induction. We shall present two different versions.

Theorem 5.1. (Two-dimensional Induction, Version 1)

Let $S(m, n)$ denote a statement involving two variables, m and n. Suppose

(1) $S(1, 1)$ is true;

(2) if $S(k, 1)$ is true for some positive integer k, then $S(k+1, 1)$ is also true;

(3) if $S(h, k)$ holds for some positive integers h and k, then $S(h, k+1)$ is also true.

Then $S(m, n)$ is true for all positive integers m, n.

Theorem 5.1 can be easily understood. The first two conditions together imply (by Theorem 2.1) that $S(m, 1)$ is true for all positive integers m. Thus, fixing m, this together with condition (3) imply (by Theorem 2.1 again) that $S(m, n)$ holds for all positive integers n. As a result, $S(m, n)$ holds for all positive integers m and n, as we desire.

Example 5.1.

Let f be a function of two variables, with $f(1, 1) = 2$ and

$$\begin{cases} f(m+1, n) = f(m, n) + 2(m+n) \\ f(m, n+1) = f(m, n) + 2(m+n-1) \end{cases}$$

for all natural numbers m and n. Prove that

$$f(m, n) = (m+n)^2 - (m+n) - 2n + 2$$

for all positive integers m and n.

Solution.

We first check that $f(1, 1) = 2 = (1+1)^2 - (1+1) - 2(1) + 2$.

Suppose $f(k, 1) = (k+1)^2 - (k+1) - 2(1) + 2 = k^2 + k$ for some positive integer k.

Then $f(k+1, 1) = f(k, 1) + 2(k+1)$

$$= (k^2 + k) + (2k + 2)$$

$$= \left[(k+1) + 1\right]^2 - \left[(k+1) + 1\right] - 2(1) + 2$$

Thus conditions (1) and (2) in Theorem 5.1 are satisfied.

Suppose $f(h, k) = (h+k)^2 - (h+k) - 2k + 2$ for some positive integers h and k.

Then
$$\begin{aligned}
f(h, k+1) &= f(h, k) + 2(h+k-1) \\
&= (h+k)^2 - (h+k) - 2k + 2 + 2(h+k) - 2 \\
&= (h+k+1)^2 - (h+k+1) - 2(k+1) + 2
\end{aligned}$$

Thus condition (3) in Theorem 5.1 is also satisfied.

It follows that $f(m, n) = (m+n)^2 - (m+n) - 2n + 2$ for all positive integers m and n.

Theorem 5.1 is essentially applying Theorem 2.1 twice. The following alternative version of the principle of two-dimensional induction in some sense reduces a two-dimensional problem into one dimension.


Theorem 5.2. (Two-dimensional Induction, Version 2)

Let $S(m, n)$ denote a statement involving two variables, m and n. Suppose

(1) $S(1, 1)$ is true;

(2) if for some positive integer $k > 1$, $S(m, n)$ is true whenever $m + n = k$, then $S(m, n)$ is true whenever $m + n = k + 1$.

Then $S(m, n)$ is true for all positive integers m, n.

To see that the two conditions guarantee $S(m, n)$ to be true for all positive integers m, n, consider the following figure:
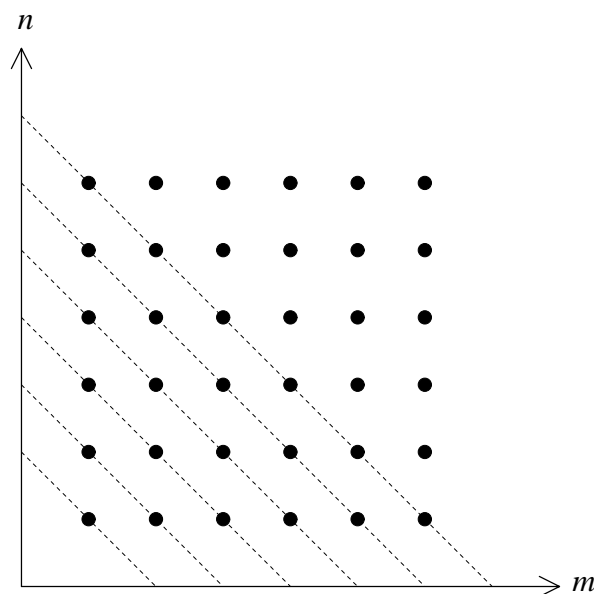
Each dotted line corresponds to a fixed value of $m+n$. It is easy to see that viewing $m+n$ as a single variable, Theorem 5.2 is essentially the same as the basic principle of mathematical induction (Theorem 2.1).

Example 5.2.

For natural numbers p and q, the Ramsey number $R(p,q)$ is defined as the smallest integer n so that among any n people, there exist p of them who know each other, or there exist q of them who don't know each other. (We assume that if A knows B, then B knows A, and vice versa.) It is known that

$$R(p,1) = R(1,q) = 1$$
$$R(p+1,q+1) \le R(p,q+1) + R(p+1,q)$$

for all natural numbers p and q. Deduce that for all natural numbers p, q,

$$R(p,q) \le C_{p-1}^{p+q-2}.$$

Solution.

First, we check that $R(1,1) = 1 = C_{1-1}^{1+1-2}$.

Assume that the desired inequality holds for all p, q with $p+q=k$, where k is a positive integer.

Now consider $R(p,q)$ with $p+q=k+1$.

If either $p=1$ or $q=1$, the desired inequality follows immediately.

If not, then noting that $(p-1)+q = p+(q-1) = k$, the inductive hypothesis gives

$$R(p,q) \le R(p-1,q) + R(p,q-1) \le C_{p-2}^{p+q-3} + C_{p-1}^{p+q-3} = C_{p-1}^{p+q-2}.$$

In other words, the desired inequality holds whenever $p+q=k+1$.

By Theorem 5.2, the result follows.

6. Jokes and Paradoxes

In this section we present some 'interesting examples' involving mathematical induction. Many of these examples have a strong 'paradox feel'. <u>It should be noted that some of the arguments are not correct</u>, but we are not going to point them out explicitly. We hope that the reader will be able to judge which arguments are correct and which are not. In case an argument is wrong, we hope the reader will be able to spot where the flaw lies.

One can drink any amount of water when feeling thirsty

We will prove the statement 'when one feels thirsty one will be able to drink n drops of water' using mathematical induction.

Clearly, the statement holds for $n = 1$ because one certainly wants to drink some water when feeling thirsty.

Suppose the statement holds for $n = k$, i.e. when one feels thirsty one is able to drink k drops of water.

Consider the case $n = k + 1$. By assumption, when one feels thirsty one is able to drink k drops of water. Being thirsty, one certainly is able to drink one more drop. So the statement holds for $n = k + 1$ as well.

By the principle of mathematical induction, the statement holds for all natural numbers n. In other words, when one feels thirsty, one can drink any amount of water. In particular, one is able to swallow up all the water in the oceans!

Everything in the world is of the same colour

We will prove the statement 'any n things in the world are of the same colour' using mathematical induction.

Clearly, the statement holds when $n = 1$ because anything has the same colour as itself.

Suppose the statement holds when $n = k$, i.e. any k things in the world are of the same colour.

Now consider the case $n = k + 1$. Take $k + 1$ things in the world and line them up in a row. By the induction hypothesis, the first k things are of the same colour. By the induction hypothesis again, the last k things are also of the same colour. Therefore, the $k + 1$ things are of the same colour.

In other words, the statement holds for $n = k + 1$ if it holds for $n = k$. By the principle of mathematical induction, the statement holds for all positive integers n. That is, everything in the world is of the same colour!

Superpower?

There are two people, each given a positive integer, such that the two integers differ by 1. The rule of the game requires that if anyone knows that if his number is the smaller among the two, then he should put up his hand. We are going to prove that the one with the smaller number will always put up his hand. (Note that this essentially means he has superpower, for suppose the two numbers are 2003 and 2004, the one given 2003 only knows that the other number is 2002 or 2004, but should not be able to tell whether it is 2002 or 2004.)

Suppose the two people are given positive integers n and $n+1$. We shall prove that the one given the integer n will put up his hand. Note that this is true when $n=1$, because 1 is the smallest positive integer, so the one given the number 1 must know that his number is the smaller among the two.

Suppose the statement holds when $n=k$. Consider the case $n=k+1$. Then the two people are given $k+1$ and $k+2$. The one with $k+1$ will think, 'The other number is either k or $k+2$. Suppose it is k, then by the induction hypothesis, the other person should put up his hand. But he doesn't, so the other number must be $k+2$.' Consequently, the one with $k+1$ will put up his hand.

By the principle of mathematical induction, the result follows.

What is the colour of my hat? (Part 1)

There are n people, each being put on a hat among not fewer than n white hats and $n-1$ black hats. They then queue up in a row, so that everyone can see only the hats of those standing in front of him. Now starting from the one at the back, we ask the question 'do you know the colour of your hat?' If the first $n-1$ people all say 'no', then the last person (the one at the front of the queue) must say 'yes'.

The case $n=1$ is trivial, since there is no black hat. Suppose the statement holds for $n=k$. Consider the case $n=k+1$. The k people at the back all say 'no', and we need to see whether the one at the front can tell the colour of his hat. The one at the front will think as follows.

'Suppose my hat is black. Then discarding me and my black hat from consideration, there are only k people with at least k white hats and $k-1$ black hats. By the induction hypothesis, since the first $k-1$ people answered "no", the one behind me must say "yes". But now he said "no". Therefore my hat cannot be black. It must be white.'

Hence the one in the front answers 'yes', i.e. the statement holds for $n=k+1$.

By the principle of mathematical induction, the statement is proved.

What is the colour of my hat? (Part 2)

Now suppose the n people can see each other (so each person knows the colour of the hat of everybody except himself), and anyone who knows the hat on his head need to put up his hand. We will prove that those with white hats will all put up their hands.

Suppose r people are put on white hats. Then for $r=1$, the one with white hat sees $n-1$ black hats. Yet there are only $n-1$ black hats. So he knows that his own hat is white, and he puts up his hand.

Now suppose our assertion is true when k people are put on white hats. Consider the case $r = k+1$. A person with white hat sees k other people in white hat and he will think, 'If my hat is black, then taking myself and my hat away from consideration, there are $n-1$ people, at least $n-1$ white hats and $n-2$ black hats. Among them, k of them are put on white hats. By the induction hypothesis, those with white hats should have put up their hands. But they don't. So my hat is not black; it is white.' Consequently, he puts up his hand. All others with white hats will do the same.

By the principle of mathematical induction, the statement is proved.

Two situations revisited

Go back to the example where two people are given two consecutive positive integers. Now instead of putting up hands at any time, suppose one can put up hands only at certain fixed time, say when a light flashes, and that the light flashes every minute.

Then, suppose the two people are given n and $n+1$, the one given the number n will put up his hand at the n-th minute. The proof is essentially the same as that given in 'Superpower' above, and we leave it as an exercise.

Similarly, in the situation of 'What is the colour of my hat? (Part 2)', we can prove that if one is allowed to put up his hand only when the light flashes, and that if r people are put on white hats, then the people with white hats will put up their hands at the r-th minute.

All professors resign

The following interesting example is modified from an exercise problem in *Calculus* by M. Spivak.

There are 17 professors in the Department of Mathematics of a certain university. They have a meeting every week. There is a rule saying that if any professor finds mistakes in his own papers, then he must resign.

For long no professor has ever resigned. This does not mean that no professor has ever made any mistake in their papers. Rather, everyone has made mistakes, and every other professor discovers that. In other words, everyone knows that every other professor has made mistakes, but does not know his own mistake.

On the last meeting of a year, a research assistant said, 'I must tell you one thing. Among the professors, at least one has made mistakes in his papers, and was discovered by someone else.'

According to the 'hat colour' example, all professors resign at the 17th meeting of the subsequent year. We leave the details as an exercise.

One interesting question is that, what the research assistant points out is something that every professor knows. Why would nobody resign without his remark, and why would every professor resign after he made this 'well-known' remark?

7. Exercises

1. Prove by mathematical induction that the following statements hold for all positive integers n.

   (a) $1^2 + 2^2 + \cdots + n^2 = \dfrac{1}{6}n(n+1)(2n+1)$

   (b) $1^2 \times 2 + 2^2 \times 3 + \cdots + n^2(n+1) = \dfrac{n(n+1)(n+2)(3n+1)}{12}$

   (c) $4007^n - 1$ is divisible by 2003.

   (d) $2002^{n+2} + 2003^{2n+1}$ is divisible by 4005.

   (e) $n^2 > n + 1$

   (f) $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{2^n} \geq \dfrac{n}{2}$

   (g) $1 \times 1! + 2 \times 2! + \cdots + n \times n! = (n+1)! - 1$

   (h) $\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \sin\dfrac{(n+1)\theta}{2}\cos\dfrac{n\theta}{2}\csc\dfrac{\theta}{2} - 1$

2. To apply the principle of mathematical induction we need to verify two conditions, namely, the statement holds for $n = 1$, and that if the statement holds for $n = k$ it also holds for $n = k+1$. Can you think of a (wrong) statement in which the second condition is satisfied but the first one is not? That is, can you construct a statement $S(n)$ such that if $S(k)$ true, then $S(k+1)$ must be true, yet $S(1)$ is not true?

3. Prove that for all natural numbers n,

   $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

   What is the significance of the above result on the convergence of the series $\sum n^{-2}$?

4. The *Lucas sequence* 1, 3, 4, 7, 11, 18, 29, … is defined by

$$a_1 = 1$$
$$a_2 = 3$$
$$a_n = a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3$$

Prove, by mathematical induction, that $a_n < (1.75)^n$ for all positive integers n.

5. From a pack of 52 playing cards one extracts the 26 red cards and pairs them up randomly. The back sides of each pair of cards are then glued together, resulting in 13 cards with both sides being 'the front'. Prove that it is always possible to flip the cards so that the 13 sides facing upward are A, 2, 3, …, 10, J, Q, K.

6. Answer the question at the end of Example 2.3.

7. Formulate a new variation of the principle of mathematical induction by combining Theorem 3.1 and Theorem 3.2.

8. Complete Example 3.5 by working out the omitted details in the last step.

9. Complete Example 4.4 by working out the omitted details in the case of two squares.

10. Complete the proofs of the two problems in 'Two situations revisited' in Section 6.

11. In the subsection entitled 'All professors resign' in Section 6, explain (using the result in the situation of 'hat colours') why all professors resign at the 17th meeting in the subsequent year. Also answer the question at the end of the subsection.

12. The Fibonacci sequence is defined as $x_0 = 0$, $x_1 = 1$ and $x_{n+2} = x_{n+1} + x_n$ for all non-negative integers n. Prove that

    (a) $x_m = x_{r+1} x_{m-r} + x_r x_{m-r-1}$ for all integers $m \geq 1$ and $0 \leq r \leq m-1$;

    (b) x_d divides x_{kd} for all positive integers d and k.

13. Among the many examples in this set of notes proven using mathematical induction, try to give another proof which does not make use of mathematical induction. (Some may not be possible or may be very difficult to prove without using induction; just try as many as you can.)