MAT 127: Calculus C, Fall 2010
Solutions to Early Exam

1. $\dfrac{2}{4^{-1}}$ *equals*

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$ (C) 2   (D) 4   (E) 8

$$\frac{2}{4^{-1}} = 2\cdot\left(4^{-1}\right)^{-1} = 2\cdot 4^{(-1)\cdot(-1)} = 2\cdot 4^{1} = 2\cdot 4 = 8.$$

2. $\dfrac{1}{4^n - 2^n}$ *equals*

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{2^n}$ (C) $\dfrac{1}{4^n} - \dfrac{1}{2^n}$ (D) $\dfrac{1}{2^n - 1} - \dfrac{1}{2^n}$ (E) $\dfrac{1}{2^n - 1}$

$$\frac{1}{2^n - 1} - \frac{1}{2^n} = \frac{2^n - (2^n - 1)}{(2^n - 1)2^n} = \frac{1}{2^n\cdot 2^n - 2^n} = \frac{1}{2^{n+n} - 2^n} = \frac{1}{(2^2)^n - 2^n} = \frac{1}{4^n - 2^n}$$

3. *The quadratic equation $x^2 - 6x - 3 = 0$ has two distinct roots. Their sum and product are*

(A) -6 and -3 (B) 6 and 3   (C) 6 and -3 (D) -6 and 3   (E) 3 and -6

The sum of the roots of the quadratic equation $x^2 + px + q = 0$ is $-p$, while their product is q. In this case, $p = -6$ and $q = -3$.

4. *The number of solutions of the system of equations* $\begin{cases} 3x + 4y = 7 \\ x + \sqrt{2}y = 3 \end{cases}$ *is*

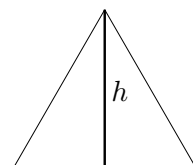(A) 0   (B) 1   (C) 2   (D) 7   (E) infinite

The system of equations

$$\begin{cases} ax + by = p \\ cx + dy = q \end{cases}$$

has a unique solution if and only if $ad \neq bc$ (otherwise, it has either no solutions or infinitely many solutions). In this case, $a = 3$, $b = 4$, $c = 1$, and $d = \sqrt{2}$; so $ad = 3\sqrt{2}$ is not a rational number, while $bc = 4$ is.

5. *The height h (length of altitude) in a triangle with all sides 1 is*

(A) $\dfrac{1}{2}$ (B) $\dfrac{\sqrt{3}}{2}$ (C) 1   (D) $\dfrac{3}{2}$ (E) $\sqrt{3}$

Since all sides in the (larger) triangle are equal, all angles are equal to $180°/3 = 60°$. Since the altitude h is opposite to a $60°$-angle in a right triangle with hypotenuse 1,

$$h = 1 \cdot \sin 60° = \frac{\sqrt{3}}{2}$$

6. *The sum* $\quad 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{1024}$ *equals*

(A) 1   (B) $\dfrac{2043}{1024}$ (C) $\dfrac{2045}{1024}$ (D) $\dfrac{2047}{1024}$ (E) $\dfrac{4095}{2048}$

This is a truncated geometric series with ratio $\frac{1}{2}$, first term 1, and first truncated term $\frac{1}{1024} \cdot \frac{1}{2} = \frac{1}{2048}$. Thus,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{1024} = \sum_{k=0}^{k=10} \frac{1}{2^k} = \frac{1 - \frac{1}{2048}}{1 - \frac{1}{2}} = \frac{2047}{1024}$$

7. $\ln 81 - \ln 36$ *equals*

(A) 0   (B) 1   (C) $\ln 45$ (D) $2\ln 3 - 2\ln 2$ (E) 45

$$\ln 81 - \ln 36 = \ln 3^4 - \ln(2^2 \cdot 3^2) = 4\ln 3 - \ln 2^2 - \ln 3^2 = 4\ln 3 - 2\ln 2 - 2\ln 3 = 2\ln 3 - 2\ln 2$$

8. $e^{2\ln 3}$ *equals*

(A) 0   (B) 1   (C) 6   (D) 8   (E) 9

$$e^{2\ln 3} = \left(e^{\ln 3}\right)^2 = 3^2 = 9$$

9. $10^{\ln 6} - 6^{\ln 10}$ *equals*

(A) 0   (B) 1   (C) -1 (D) $4^{-\ln 4}$ (E) $\dfrac{1}{256}$

$$10^{\ln 6} - 6^{\ln 10} = \left(e^{\ln 10}\right)^{\ln 6} - \left(e^{\ln 6}\right)^{\ln 10} = e^{(\ln 10)(\ln 6)} - e^{(\ln 6)(\ln 10)} = 0.$$

10. $\cos x + \cos(-x)$ *equals*

(A) 0   (B) 1   (C) −1   (D) $2\cos x$ (E) $\cos 2x$

Since $\cos(-x) = \cos x$, $\cos x + \cos(-x) = 2\cos x$.

11. *For every number $x > 0$, the values of x, e^x, and $\ln(x)$ are all different. Which list has them ordered from smallest to largest?*

(A) x, e^x, $\ln(x)$ (B) x, $\ln(x)$, e^x (C) $\ln(x)$, x, e^x (D) $\ln(x)$, e^x, x (E) e^x, x, $\ln(x)$

Since $e^x = 1 + x + \dfrac{x^2}{2!} + \dots$, $e^x > x$. Since $\ln x$ is the inverse function of e^x, $\ln x < x$ (the graph of $y = \ln x$ is the reflection about the line $y = x$ of the graph of $y = e^x$).

12. *The graph of the function $y = (x-2)^3 + 3$ is obtained by shifting the graph of the function $y = x^3$*

(A) 3 units up and 2 units left
(B) 3 units up and 2 units right
(C) 3 units down and 2 units left
(D) 3 units down and 2 units right
(E) 3 units right and 2 units down

The graph of $y = x^3$ passes through $(0,0)$, while the graph of $y = (x-2)^3 + 3$ passes through $(2, 3)$.

13. $\displaystyle\lim_{x \to 0} \dfrac{\tan x}{x}$ *equals*

(A) 0   (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{2}$ (D) 1   (E) ∞

By the product rule for limits and l'Hospital rule,

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{1}{\cos x} \cdot \lim_{x \to 0} \frac{\sin x}{x} = 1 \cdot \lim_{x \to 0} \frac{(\sin x)'}{x'} = \lim_{x \to 0} \frac{\cos x}{1} = 1;$$

l'Hospital rule is applicable here because $\sin x, x \longrightarrow 0$ as $x \longrightarrow 0$.

14. $\displaystyle\lim_{x \to \infty} \dfrac{x^2}{1 + \sqrt{4x^4 + 1}}$ *equals*

(A) 0   (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{2}$ (D) 1   (E) ∞

Divide the numerator and denominator by x^2:

$$\frac{x^2}{1 + \sqrt{4x^4 + 1}} = \frac{1}{1/x^2 + \sqrt{4x^4 + 1}/\sqrt{x^4}} = \frac{1}{1/x^2 + \sqrt{(4x^4 + 1)/x^4}}$$

$$= \frac{1}{1/x^2 + \sqrt{4 + 1/x^4}} \xrightarrow[x \to \infty]{} \frac{1}{0 + \sqrt{4 + 0}} = \frac{1}{2}$$

15. $\lim\limits_{x \longrightarrow \infty} \dfrac{\sin x}{x}$ *equals*

(A) 0   (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{2}$ (D) 1   (E) ∞

Since $|\sin(x)| \leq 1$, $|\sin x|/x \longrightarrow 0$ as $x \longrightarrow \infty$. l'Hospital rule is *not* applicable here because the denominator of the fraction approaches ∞ as $x \longrightarrow \infty$, while the numerator does not.

16. *Which of the following equations describes the line tangent to the graph of the function*

$$f(x) = (\sin x)e^{\cos x} \qquad at \ (\pi/2, 1) \ ?$$

(A) $y = -x + \frac{\pi}{2} + 1$ (B) $y = -x - \frac{\pi}{2} + 1$ (C) $y = x - \frac{\pi}{2} + 1$ (D) $y = x - \frac{\pi}{2}$ (E) $y = x + 1$

By the product and chain rules for differentiation,

$$f'(x) = (\sin x)' \cdot e^{\cos x} + (\sin x) \cdot \left(e^{\cos x}\right)' = (\cos x)e^{\cos x} + (\sin x) \cdot \left(e^{\cos x} \cdot (-\sin x)\right)$$
$$= (\cos x)e^{\cos x} - (\sin^2 x)e^{\cos x} .$$

Thus, $f'(\pi/2) = -1$ and the tangent line is described by the equation

$$y - y_0 = (-1)(x - x_0) \qquad with \qquad (x_0, y_0) = (\pi/2, 1).$$

17. $\displaystyle\int \sin(2x)\,\mathrm{d}x$ *equals*

(A) $\cos 2x + C$ (B) $-\cos 2x + C$ (C) $\dfrac{1}{2}\cos 2x + C$ (D) $-\dfrac{1}{2}\cos 2x + C$ (E) $-2\cos(2x) + C$

Use the substitution $u = 2x$, so that $\mathrm{d}x = \mathrm{d}u/2$ and

$$\int \sin(2x)\,\mathrm{d}x = \int \sin(u)\,\frac{\mathrm{d}u}{2} = \frac{1}{2}(-\cos u + C) = -\frac{1}{2}\cos 2x + C'$$

18. $\displaystyle\int_0^\infty x\mathrm{e}^{-2x}\,\mathrm{d}x$ *equals*

(A) 0   (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2}$ (E) 1

Use integration by parts:

$$\int_0^\infty x\mathrm{e}^{-2x}\,\mathrm{d}x = -\frac{1}{2}\int_0^\infty x\mathrm{d}\mathrm{e}^{-2x} = -\frac{1}{2}\left(x\mathrm{e}^{-2x}\Big|_0^\infty - \int_0^\infty \mathrm{e}^{-2x}\,\mathrm{d}x\right)$$
$$= -\frac{1}{2}\left((0 - 0) - \left(-\frac{1}{2}\mathrm{e}^{-2x}\Big|_0^\infty\right)\right) = -\frac{1}{4}(0 - \mathrm{e}^{2\cdot0}) = \frac{1}{4}$$

19. $\displaystyle\int_0^5 \frac{1}{x^2 + 4x + 3}\,\mathrm{d}x$ *equals*

(A) $\dfrac{1}{2}$ (B) $\dfrac{5}{16}$ $\boxed{\text{(C)}}$ $\ln 3 - \ln 2$ (D) $\ln 2$ (E) $\arctan 7 - \arctan 2$

Use (*quick*) **partial fractions** to break the fraction into two with *linear* denominators:

$$\frac{1}{x^2 + 4x + 3} = \frac{1}{(x+1)(x+3)} = \frac{1}{\boxed{+3} - (\boxed{+1})}\left(\frac{1}{x\boxed{+1}} - \frac{1}{x\boxed{+3}}\right) = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x+3}\right)$$

This *quick partial fractions* approach gives the answer *immediately* without solving for the coefficients as in Section 5.7. QPF works in this case because the coefficients of x in the two factors are the same (both are 1 here); it would not have worked if they were different or if *one* of the x's were replaced by something else, such as x^3 (you would have to use the approach of Section 5.7 then). It does not matter in what order the two fractions are written in the third expression, as long as the constant terms are copied to the denominator of the external fraction **without the arrows crossing**.

We can now compute the integral:

$$\int_0^5 \frac{1}{x^2 + 4x + 3}\,\mathrm{d}x = \frac{1}{2}\int_0^5 \left(\frac{1}{x+1} - \frac{1}{x+3}\right)\mathrm{d}x = \frac{1}{2}\Big(\ln(x+1) - \ln(x+3)\Big)\Big|_0^5$$

$$= \frac{1}{2}\big((\ln 6 - \ln 8) - (\ln 1 - \ln 3)\big) = \frac{1}{2}\ln\left(\frac{6 \cdot 3}{8}\right) = \frac{1}{2}\ln\left(\frac{3}{2}\right)^2$$

$$= \ln\left(\frac{3}{2}\right) = \ln 3 - \ln 2$$

Partial fractions will be useful for computing some integrals in Chapters 7 and 8 and sums of some infinite series by telescoping cancellation in Chapter 8.

20.   *Which of the following statements is true?*

(A) $\displaystyle\int_2^{12} x^{-3/2}\mathrm{d}x < \sum_{n=3}^{n=12} n^{-3/2} < \sum_{n=2}^{n=11} n^{-3/2}$ $\boxed{\text{(B)}}$ $\displaystyle\sum_{n=3}^{n=12} n^{-3/2} < \int_2^{12} x^{-3/2}\mathrm{d}x < \sum_{n=2}^{n=11} n^{-3/2}$

(C) $\displaystyle\sum_{n=3}^{n=11} n^{-3/2} < \int_2^{12} x^{-3/2}\mathrm{d}x < \sum_{n=3}^{n=12} n^{-3/2}$ (D) $\displaystyle\sum_{n=2}^{n=12} n^{-3/2} < \sum_{n=1}^{n=11} n^{-3/2} < \int_2^{12} x^{-3/2}\mathrm{d}x$

(E) $\displaystyle\sum_{n=2}^{n=12} n^{-3/2} < \int_2^{12} x^{-3/2}\mathrm{d}x < \sum_{n=1}^{n=11} n^{-3/2}$

Since $f(x) = x^{-3/2}$ is a positive function (for $x > 0$), $\displaystyle\int_2^{12} x^{-3/2}\mathrm{d}x$ is the area under the graph of f between the vertical lines $x=2$ and $x=12$. As any integral, it can be estimated using the right end points of the unit intervals $[2, 3], [3, 4], \ldots, [11, 12]$; see Sections 5.1 and 5.9. This estimate is the sum of the areas of the rectangles of base 1 ($= 3-2, 4-3, \ldots$) and heights $f(3), f(4), \ldots, f(12)$:

$$1 \cdot f(3) + 1 \cdot f(4) + \ldots + 1 \cdot f(12) = \sum_{n=3}^{n=12} n^{-3/2}.$$

Since f is a decreasing function, this sum is smaller than the area under the graph, as indicated in the first figure below; this gives the first inequality in (B). This integral can also be estimated using the left end points of the same intervals as

$$1 \cdot f(2) + 1 \cdot f(3) + \ldots + 1 \cdot f(11) = \sum_{n=2}^{n=11} n^{-3/2}.$$

Since f is a decreasing function, this sum is larger than the area under the graph, as indicated in the second figure below; this gives the second inequality in (B).

In (A), the first inequality is wrong (being opposite to the first inequality in (B)), though the second is correct ($12^{-3/2}$ in the middle expression is replaced by $2^{-3/2} > 12^{-3/2}$ in the last expression). In (C), the second inequality is wrong (being the same as the first inequality in (A)), though the first is correct (the first sum in (C) is even smaller than the first sum in (B), because the former is missing $12^{-3/2} > 0$). In (D), the second inequality is wrong (being even stronger than the opposite of the second inequality in (B), because it has an extra $1^{-3/2} = 1 > 0$), though the first is correct ($12^{-3/2}$ in the first expression is replaced by $1^{-3/2} = 1 > 12^{-3/2}$ in the middle expression). In (E), the first inequality is wrong (being even stronger than the opposite of the second inequality in (B), because it has an extra $12^{-3/2} > 0$), though the second is correct (the last sum in (E) is even larger than the last sum in (B), because the former has an extra $1^{-3/2} = 1 > 0$).
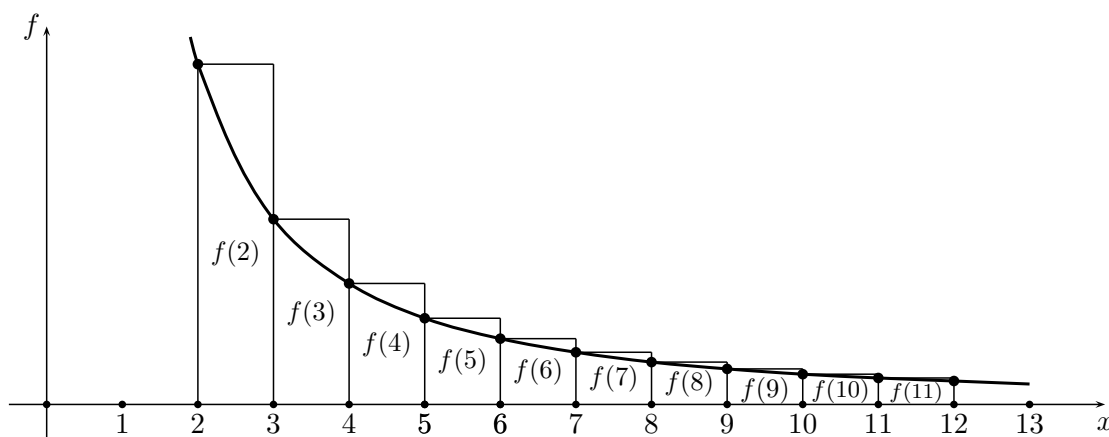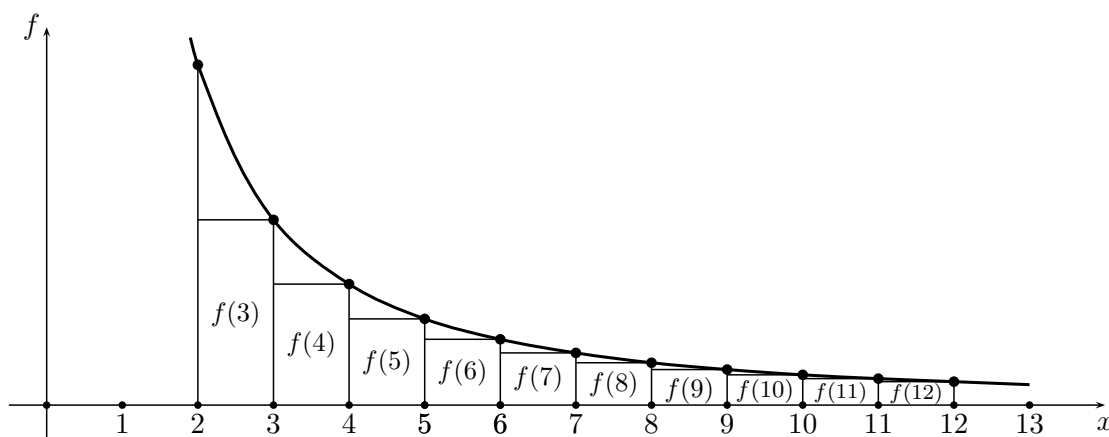
An "infinite" version of this question is the basis behind the *Integral Test for Series* in Section 8.3.