MAT 127: Calculus C, Spring 2017
Notes on the Ratio Test: for Sequences and Series

0 Introduction

In the textbook, the *Ratio Test* is introduced in Section 8.4 as one of many convergence tests for *series*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots .$$

A convergence test for *sequences* $a_1, a_2, \ldots$ can be deduced from *RT for Series* using the *Divergence Test for Series* (box 6 or 7 on p570). However, this convergence test can also be obtained more directly. Just like *RT for Series*, *RT for Sequences* works well for some sequences and is useless for others. On the other hand, *RT for Series* is perhaps the most frequently used convergence test for series, while *RT for Sequences* is perhaps the least frequently used convergence test for series (but it does work very nicely in some cases!).

In order to use either of the ratio tests, look at the ratio of the absolute values of two consecutive terms, $|a_{n+1}|/|a_n|$; note that the higher-numbered term goes to the top of this fraction and the lower-numbered term goes to the bottom. For example, for the sequence $a_n = (-1)^n 7^n/n!$, this ratio is

$$\frac{|a_{n+1}|}{|a_n|} = \frac{7^{n+1}/(n+1)!}{7^n/n!} = \frac{7^{n+1}}{7^n} \cdot \frac{n!}{(n+1)!} = \frac{7^n \cdot 7}{7^n} \cdot \frac{n!}{n! \cdot (n+1)} = \frac{7}{n+1} .$$

This produces a new sequence $a_2/a_1, a_3/a_2, a_4/a_3, \ldots$, which consists of non-negative terms.

1 Ratio Test for Sequences

First, suppose we want to determine whether the *sequence* $a_1, a_2, \ldots$ has a limit. We could instead look at the sequence of the absolute values of the ratios of consecutive terms $|a_{n+1}|/|a_n|$. If the latter sequence has a limit L (which must necessarily be non-negative) and

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} < 1, \qquad \textbf{then} \qquad \lim_{n\longrightarrow\infty} a_n = 0. \tag{1}$$

On the other hand, if

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \quad \text{or} \quad \frac{|a_{n+1}|}{|a_n|} \longrightarrow \infty, \qquad \textbf{then} \qquad |a_n| \longrightarrow \infty, \tag{2}$$

and so the sequence a_n diverges (or possibly “converges” to infinity). Finally, if

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 1, \qquad \textbf{then} \qquad \text{this test says nothing.} \tag{3}$$

In this last case, you’ll need to find some other way to determine if the sequence $a_1, a_2, \ldots$ converges.

For example, for the sequence $a_n = (-1)^n 7^n / n!$,

$$\lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \longrightarrow \infty} \frac{7}{n+1} = \frac{7}{\infty + 1} = 0.$$

Since $0 < 1$, this sequence converges to 0. On the other hand, for the sequence $a_n = 2^n / n$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}/(n+1)}{2^n/n} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2^n \cdot 2}{2^n} \cdot \frac{1}{n/n + 1/n} = 2\frac{1}{1+1/n} \longrightarrow 2\frac{1}{1+1/\infty} = 2\frac{1}{1+0} = 2.$$

Since $2 > 1$, this sequence diverges (actually "converges" to ∞).

For the sequence $1, 1, 1, \ldots$, the limit of the ratios of the absolute values of consecutive terms is 1 (all of these ratios are 1) and this sequence converges 1. For the sequence $-1, 1, -1, 1 \ldots$, the limit of the ratios of the absolute values of consecutive terms is also 1 (all of these ratios are again 1), but this sequence diverges (it keeps on jumping between 1 and -1). This shows that *RT for Sequences* is useless in the case

$$L \equiv \lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = 1.$$

Even a sequence $a_1, a_2, a_3, \ldots$ with $L = 1$ *and* $a_n > 0$ need not converge. For example, let

$$a_1 = 2^{\frac{1}{1}}, \quad a_2 = 2^{\frac{1}{1}+\frac{1}{2}}, \quad a_3 = 2^{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}}, \quad \ldots . \tag{4}$$

Then,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}+\frac{1}{n+1}}}{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}} = \frac{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}} \cdot 2^{\frac{1}{n+1}}}{2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}}} = 2^{\frac{1}{n+1}} \longrightarrow 2^{\frac{1}{\infty+1}} = 2^0 = 1.$$

However,

$$2^{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}} \longrightarrow 2^{\infty} = \infty,$$

because the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$$

diverges by the *p-Series Test* on p578 (this is $p = 1$; see also Example 7 on p569). Thus, the sequence of positive terms $a_1, a_2, \ldots$ described in (4) diverges (it actually "converges" to ∞), even though $L = 1$ in this case.

Why is (1) true? The assumption in (1) is that the ratios $|a_{n+1}|/|a_n|$ get very close to L as n increases. Since $L < 1$ in this case, this means that $|a_{n+1}|/|a_n| < (L+1)/2$ if n is very large, so that

$$|a_{n+1}| < \frac{L+1}{2} |a_n|$$

for all n larger than some N. Thus,

$$|a_{N+n}| < \frac{L+1}{2} |a_{N+n-1}| < \frac{L+1}{2} \cdot \frac{L+1}{2} |a_{N+n-2}| \ldots < \underbrace{\frac{L+1}{2} \cdot \ldots \cdot \frac{L+1}{2}}_{n} |a_N| = \left(\frac{L+1}{2}\right)^n |a_N|.$$

Thus, the sequence $|a_N|, |a_{N+1}|, \ldots$ is squeezed between the sequence $0, 0, \ldots$ and the geometric sequence

$$|a_N|,\ \frac{L+1}{2}|a_N|,\ \left(\frac{L+1}{2}\right)^2|a_N|,\ \ldots$$

This geometric sequence converges to 0 because $|(L+1)/2| < 1$ in this case (see box 7 on p560). Thus, the sequence $|a_N|, |a_{N+1}|, \ldots$ also converges to 0 by the *Squeeze Theorem for Sequences* on p557 and so does the sequence $a_N, a_{N+1}, \ldots$ by Theorem 4 on p557. Since the convergence of a sequence has nothing to do with how it begins, it follows that the original sequence $a_1, a_2, \ldots$ also converges.

Why is (2) true? The assumption in (2) is that the ratios $|a_{n+1}|/|a_n|$ become larger than some number $r > 1$ as n increases (in the first case, we can take $r = (L+1)/2$; in the second case, r can be taken to be *any* number larger than 1). Thus, $|a_{n+1}| > r|a_n|$ for all n larger than some N and so

$$\big|a_{N+n}\big| > r\big|a_{N+n-1}\big| > r \cdot r\big|a_{N+n-2}\big| \ldots > \underbrace{r \cdot \ldots \cdot r}_{n}\big|a_N\big| = r^n|a_N|.$$

So, the terms in the sequence $|a_N|, |a_{N+1}|, \ldots$ are larger than the terms in the geometric sequence

$$|a_N|,\ r|a_N|,\ r^2|a_N|,\ \ldots$$

This geometric sequence diverges (actually "converges" to ∞) because $r > 1$ (see box 7 on p560). Thus, the sequence $|a_N|, |a_{N+1}|, \ldots$ also diverges (also "converges" to ∞), since its terms are even larger. Since the convergence of a sequence has nothing to do with how it begins, it follows that the original sequence $a_1, a_2, \ldots$ also diverges.

Because of (1), (2), and (3), *RT for Sequences* can detect convergence of sequences $a_1, a_2, \ldots$ with limit 0 only (and even of only some of these). So it can rarely be used to detect convergent sequences, but whenever it is applicable, *RT for Sequences* determines the limit of convergent sequences immediately. *RT for Sequences* has a good chance of working for sequences that involve factorials and powers n (e.g. $n!$, 3^n, n^n), but has *no* chance of working for sequences that involve only powers of n (e.g. n^3).

2 Ratio Test for Series

RT for Series works similarly to *RT for Sequences* to determine whether certain series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

converge. As before, we look at the sequence of the absolute values of the ratios of consecutive terms $|a_{n+1}|/|a_n|$ (still sequence, not series!). If this sequence has a limit L (which must necessarily be non-negative) and

$$L \equiv \lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1, \qquad \textbf{then} \qquad \sum a_n \ \text{ converges.} \tag{5}$$

On the other hand, if

$$L \equiv \lim_{n \longrightarrow \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \quad \text{or} \quad \frac{|a_{n+1}|}{|a_n|} \longrightarrow \infty, \qquad \textbf{then} \qquad \sum a_n \ \text{ diverges.} \tag{6}$$

Finally, if

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 1, \qquad \textbf{then} \qquad \text{this test says nothing.} \tag{7}$$

In this last case, you'll need to find some other way to determine if the series $\sum a_n$ converges.

For example, for the series $\sum (-1)^n \frac{7^n}{n!}$, $a_n = (-1)^n 7^n/n!$ and thus

$$\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\longrightarrow\infty} \frac{7}{n+1} = \frac{7}{\infty+1} = 0.$$

Since $0 < 1$, this series converges. We will see at the end of the course, using *power* series, that

$$\sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!} \equiv 1 - \frac{7}{1!} + \frac{7^2}{2!} - \frac{7^3}{3!} + \ldots = \mathrm{e}^{-7}.$$

Try this on your calculator: as you add/subtract more and more of the terms, the total should approach e^{-7}.

On the other hand, for the series $\sum 2^n/n$, $a_n = 2^n/n$ and thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}/(n+1)}{2^n/n} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2^n \cdot 2}{2^n} \cdot \frac{1}{n/n + 1/n} = 2\frac{1}{1+1/n} \longrightarrow 2\frac{1}{1+1/\infty} = 2\frac{1}{1+0} = 2.$$

Since $2 > 1$, this series diverges. In fact, if you are asked to determine whether some series $\sum a_n$ converges or diverges, you should *always* begin by applying the *Divergence Test for Series* (box 6 or 7 on p570) first, i.e. testing whether the *sequence* $a_1, a_2, \ldots$ converges to 0 (if this sequence does not converge to 0, the series do not converge at all, but **the converse need not hold**). So, if you are asked to determine whether the series

$$\sum_{n=1}^{\infty} 2^n/n = \frac{2^1}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \ldots$$

converges, first check whether the *sequence* $a_n = 2^n/n$ converges to 0. Since this sequence involves an exponential of n, try *RT for Sequences*. Since $|a_{n+1}/a_n| \longrightarrow 2$ by the above, by (2) this sequence does not converge at all, so the corresponding series diverges as well.

Remark: If you find that the sequence $a_1, a_2, \ldots$ does converge to 0, then you **cannot conclude** that the series $\sum a_n$ converges and must use one of the many other convergence tests for series (that is why they are there; otherwise, they would not be needed). In the case of the series,

$$\sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!} \equiv 1 - \frac{7}{1!} + \frac{7^2}{2!} - \frac{7^3}{3!} + \ldots,$$

$|a_{n+1}/a_n| \longrightarrow 0$ by the above. So the sequence $|a_n|$ converges to 0 by (1), and you must then find another convergence test to determine if the *series* itself converges. In this case, you can then apply *RT for Series* to conclude that the series converges, having already computed the limit L of the ratios of the absolute values of consecutive terms. So, for the series $\sum a_n$ to which *RT for Series* is applicable, there is no harm in forgetting to start with the *Divergence Test for Series* (box 6 or 7 on p570). However, always starting with the *Divergence Test for Series* provides a systematic approach to analyzing whether a series converges.

The sequence $a_n = 1/n^p$ converges to 0 if $p>0$. However, by the *p-Series Test* on p578, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

converges if and only if $p>1$. No matter what p is,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/(n+1)^p}{1/n^p} = \left(\frac{n}{n+1}\right)^p = \left(\frac{1}{1+1/n}\right)^p \longrightarrow \left(\frac{1}{1+1/\infty}\right)^p = 1.$$

This shows that *RT for Series* is useless in the case

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 1,$$

and you have to find some other convergence test to use.

Why is (5) true? The assumption in (5) is that the ratios $|a_{n+1}|/|a_n|$ get very close to L as n increases. Since $L<1$ in this case, this means that $|a_{n+1}|/|a_n| < (L+1)/2$ if n is very large, so that

$$|a_{n+1}| < \frac{L+1}{2}|a_n|$$

for all n larger than some N. Thus,

$$\big|a_{N+n}\big| < \frac{L+1}{2}\big|a_{N+n-1}\big| < \frac{L+1}{2}\cdot\frac{L+1}{2}\big|a_{N+n-2}\big| \dots < \underbrace{\frac{L+1}{2}\cdot\ldots\cdot\frac{L+1}{2}}_{n}\big|a_N\big| = \left(\frac{L+1}{2}\right)^n |a_N|.$$

Since $(L+1)/2 < 1$ in this case, the geometric series

$$\sum_{n=0}^{\infty}\left(\frac{L+1}{2}\right)^n |a_N| = |a_N| \sum_{n=0}^{\infty}\left(\frac{L+1}{2}\right)^n$$

converges by box 4 on p567. Thus, the series of positive terms

$$\sum_{n=0}^{\infty} |a_{N+n}| = |a_N| + |a_{N+1}| + |a_{N+2}| + \dots$$

also converges by the *Comparison Test* on p579. Since the convergence of a series has nothing to do with how it begins, it follows that the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \dots$$

also converges. By the *Absolute Convergence Theorem* (box 1 on p588), this implies that the original series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

also converges (making some terms negative and some positive cannot increase the absolute value of the sum).

Why is (6) true? In this case, the *sequence* a_n does not converge to 0 (in fact, does not converge at all) by (2). Thus, the series $\sum a_n$ diverges by the *Divergence Test for Series* (box 6 or 7 on p570).

Similarly to *RT for Sequences*, *RT for Series* is applicable to only some series. Unlike *RT for Sequences*, it is applicable in *many* non-trivial cases that cannot be resolved with the *Divergence Test for Series* (box 6 or 7 on p570). Similarly to *RT for Sequences*, *RT for Series* has a good chance of working for series that involve factorials and powers n (e.g. $n!$, 3^n, n^n), but has *no* chance of working for series that involve only powers of n (e.g. n^3).

3 Summary of Ratio Tests

RT for Sequences $a_1, a_2, a_3 \ldots$ and *RT for Series* $a_1 + a_2 + a_3 + \ldots$ involve studying the sequence of the ratios of the absolute values of consecutive terms, $|a_{n+1}|/|a_n|$. Depending on what happens with this last sequence, we *may* be to conclude that the original sequence/series converges or diverges. This is summarized in the table below.

If	**then**	the sequence $a_1, a_2, \ldots$	the series $a_1 + a_2 + \ldots$
$\lim_{n\longrightarrow\infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1$		converges to 0	converges
$\lim_{n\longrightarrow\infty} \frac{\|a_{n+1}\|}{\|a_n\|} > 1$ or $\frac{\|a_{n+1}\|}{\|a_n\|} \longrightarrow \infty$		diverges	diverges
$\lim_{n\longrightarrow\infty} \frac{\|a_{n+1}\|}{\|a_n\|} = 1$		**?**	**?**

The two ratio tests work extremely well in some cases and not at all in others. They have a good chance of working for series that involve factorials and powers n (e.g. $n!$, 3^n, n^n), but *no* chance of working for series that involve only powers of n (e.g. n^3).

4 Remainder Estimate

All convergence tests for series implicitly come with a "reminder estimate". These describe how close the first few terms of a convergent series come to the sum of the entire series, i.e. how small the sum of the rest of the terms is. These remainder estimates can be useful for estimating the sum of a convergent series with specified precision (e.g. within .000001). The book explicitly states remainder estimates for two convergence tests:

(1) the *Integral Test* (box 3 on p581). This is useful for series $\sum a_n$ with $a_n = f(n)$, where f is a continuous, positive, decreasing function on some interval (a, ∞) with $f(x) \longrightarrow 0$ as $x \longrightarrow \infty$, such as $f(x) = x^{-3/2}$.

(2) the *Alternating Series Test* (p587). This is useful for series $\sum a_n$ with a_n alternating in sign and with $|a_n|$ monotonically decreasing to 0, such as $a_n = (-1)^n/n$.

While these series include many important ones, not all series are of these two forms. For example,

$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\,. \tag{8}$$

Another particularly interesting example is the series in 8.4 42 which can be used to estimate π:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}\,.$$

RT for Series can be used to show that each of these two series converges. It can also be used to determine how many terms at the beginning of the series are needed to compute the infinite sum with specified precision, i.e. make the sum of the remainder of the series smaller than the desired precision.

Suppose we are given a series $\sum a_n$ and

$$L \equiv \lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} < 1.$$

In particular, the series converges by *RT for Series*. Since the ratios $|a_{n+1}|/|a_n|$ get arbitrary close to $L<1$, for any given $r \in (L,1)$, such as $r=(L+1)/2$, we can choose a large number N so that $|a_{n+1}|/|a_n| \le r$ whenever $n \ge N$. Thus, whenever $k \ge 0$

$$|a_{N+k}| \le r|a_{N+k-1}| \le r^2|a_{N+k-2}| \le r^k|a_N|,$$

and so for any $n \ge 0$

$$\left|\sum_{k=n}^{\infty} a_{N+k}\right| \le \sum_{k=n}^{\infty} |a_{N+k}| < \sum_{k=n}^{\infty} r^k|a_N| = \frac{|a_N|}{1-r} r^n.$$

Thus, the absolute value of the sum $a_{N+n} + a_{N+n-1} + \dots$ is at most $|a_N|r^n/(1-r)$, and so the sum of the finitely many preceding terms is less than $|a_N|r^n/(1-r)$ away from the infinite sum.

For example, in the case of the infinite series (8), $a_n = 1/n!$ and so

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/(n+1)!}{1/n!} = \frac{n!}{(n+1)!} = \frac{n!}{n!\cdot(n+1)} = \frac{1}{n+1} \longrightarrow \frac{1}{\infty+1} = 0\,;$$

so $L=0$ and the series converges by *RT for Series*. If we take $r=(L+1)/2=1/2$, then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1}{n+1} \le \frac{1}{2}$$

whenever $n \ge N = 1$, and

$$\left|\sum_{k=n}^{\infty} \frac{1}{(1+k)!}\right| < \frac{|a_1|}{1-r} r^n = \frac{1}{1-1/2} \cdot 2^{-n} = \frac{1}{2^{n-1}} \qquad \Longrightarrow \qquad \mathrm{e} - \sum_{k=0}^{k=n} \frac{1}{n!} < \frac{1}{2^{n-1}}\,. \tag{9}$$

In fact, we can get a much better estimate for the remainder

$$\mathrm{e} - \sum_{k=0}^{k=n} \frac{1}{n!} = \sum_{k=n+1}^{\infty} \frac{1}{k!}$$

by using $r = 1/(n+2)$. Since $(1/(k+1)!)/(1/k!) \leq 1/(n+2)$ whenever $k \geq (n+1)$, this gives

$$\sum_{k=n+1}^{\infty} \frac{1}{k!} \leq \sum_{k=0}^{\infty} \frac{1}{(n+1)!} \cdot \left(\frac{1}{n+2}\right)^k = \frac{1}{(n+1)!} \cdot \frac{1}{1-\frac{1}{n+2}} = \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1} = \frac{n+2}{(n+1)\cdot(n+1)!}.$$

So we find that

$$0 < \mathrm{e} - \sum_{k=0}^{k=n} \frac{1}{n!} < \frac{n+2}{(n+1)\cdot(n+1)!}\,. \tag{10}$$

For example, taking $n=6$ (first seven terms in (8)), we find that

$$0 < \mathrm{e} - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{5!} + \frac{1}{6!}\right) < \frac{8}{7\cdot 7!} \qquad \text{or} \qquad 0 < \mathrm{e} - \frac{1957}{720} < \frac{1}{4410}\,.$$

The estimate (10) is much better than (9) because

$$\frac{(n+2)/((n+1)\cdot(n+1)!)}{1/2^{n-1}} \longrightarrow 0$$

by (1).

5 Concluding Remark (beyond the scope of the course)

The assumptions in (1) and (5) can be weakened, without changing the conclusion. The only property of the sequence $|a_{n+1}|/|a_n|$ used in the justification of these statements above is that $|a_{n+1}|/|a_n| < r$ for some number $r \in (0,1)$ and for all n larger than some number N. So the sequence $|a_{n+1}|/|a_n|$ must eventually stay in the interval $[0,r]$ with $r < 1$. It does not matter whether it has an actual limit in this interval or keeps on jumping between some numbers in the interval. This means that the sequence of ratios contains no subsequence going off to infinity and the limit of any convergent subsequence is less than 1. The largest of these limits of subsequences is called lim sup; it must exist even if there is no actual limit of the sequence. The assumptions in (1) and (5) can be weakened to requiring that

$$\limsup \frac{|a_{n+1}|}{|a_n|} < 1, \tag{11}$$

instead of requiring that the sequence $|a_{n+1}|/|a_n|$ converge to a number less than 1.

For example, the ratios of consecutive terms for the sequence

$$1,\ \frac{1}{2},\ \frac{1}{2\cdot 3},\ \frac{1}{2\cdot 3\cdot 4},\ \frac{1}{2\cdot 3\cdot 4\cdot 2},\ \frac{1}{2\cdot 3\cdot 4\cdot 2\cdot 3},\ \frac{1}{2\cdot 3\cdot 4\cdot 2\cdot 3\cdot 4},\ \frac{1}{2\cdot 3\cdot 4\cdot 2\cdot 3\cdot 4\cdot 2},\ \ldots$$

cycle between 1/2, 1/3, and 1/4. The largest of these numbers is 1/2, and so

$$\limsup \frac{|a_{n+1}|}{|a_n|} = \frac{1}{2}\,.$$

Since $1/2 < 1$, the original sequence converges to 0, and the corresponding series (sum of the terms in this sequence) also converges (to 40/23; why?).

The assumptions in (2) and (6) can be weakened similarly. The only property of the sequence $|a_{n+1}|/|a_n|$ used in the justification of these statements above is that $|a_{n+1}|/|a_n| > r$ for some number $r>1$ and for all n larger than some number N. So the sequence $|a_{n+1}|/|a_n|$ must eventually stay in the interval (r,∞) with $r>1$. So either there is a subsequence going off to infinity (in which case its limit is considered to be ∞) or a subsequence converging to some number larger than 1. The smallest of these limits of subsequences is called lim inf; it must exist even if there is no actual limit of the sequence. The assumptions in (2) and (6) can be weakened to requiring that

$$\liminf \frac{|a_{n+1}|}{|a_n|} > 1, \tag{12}$$

instead of requiring that the sequence $|a_{n+1}|/|a_n|$ go off to infinity or converge to a number greater than 1.

For example, the ratios of consecutive terms for the sequence

$$1,\ 2,\ 2\cdot3,\ 2\cdot3\cdot4,\ 2\cdot3\cdot4\cdot2,\ 2\cdot3\cdot4\cdot2\cdot3,\ 2\cdot3\cdot4\cdot2\cdot3\cdot4, 2\cdot3\cdot4\cdot2\cdot3\cdot4\cdot2,\ \ldots$$

cycle between 2, 3, and 4. The smallest of these numbers is 2, and so

$$\liminf \frac{|a_{n+1}|}{|a_n|} = 2\,.$$

Since $2 > 1$, the original sequence diverges, as does the corresponding series (sum of the terms in this sequence).

Since lim inf of any sequence is not larger than lim sup (the two are equal if and only if the entire sequence converges), the only remaining possibility, in addition to (11) and (12), is

$$\liminf \frac{|a_{n+1}|}{|a_n|} \le 1 \le \limsup \frac{|a_{n+1}|}{|a_n|}\,.$$

In this case, the ratio tests say nothing and you need to find another test to use.

The above refinements of the original ratio tests are summarized in the table below. Unlike the table in Section 3, it covers all possibilities. You can learn more about lim inf and lim sup in MAT 320, if you do well in this course.

If	**then**	the sequence $a_1, a_2, \ldots$	the series $a_1 + a_2 + \ldots$
$\limsup \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} < 1$		converges to 0	converges
$1 < \liminf \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert}$		diverges	diverges
$\liminf \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} \le 1 \le \limsup \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert}$		**?**	**?**