MAT 127: Calculus C, Fall 2009
Solutions to Problem Set 3

Section 7.3, Problem 3 (10pts)

Solve the differential equation

$$(x^2+1)y' = xy. \tag{1}$$

This is a separable equation. Write $y' = \mathrm{d}y/\mathrm{d}x$, move everything involving y to LHS and everything involving x to RHS, and integrate:

$$\begin{aligned}(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} = xy \quad &\Longleftrightarrow \quad \frac{\mathrm{d}y}{y} = \frac{x}{x^2+1}\mathrm{d}x \\ &\Longleftrightarrow \quad \int \frac{\mathrm{d}y}{y} = \int \frac{x}{x^2+1}\mathrm{d}x \\ &\Longleftrightarrow \quad \ln|y| = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C,\end{aligned}$$

where C is any constant. Exponentiating both sides, we obtain

$$|y| = \mathrm{e}^{\ln\sqrt{x^2+1}+C} = \mathrm{e}^{\ln\sqrt{x^2+1}} \cdot e^C = A\sqrt{x^2+1} \quad \Longleftrightarrow \quad y = \pm A\sqrt{x^2+1}; \tag{2}$$

since C is any constant, $A = e^C$ is any *positive* constant. However, we divided both sides of (1) by y. This is 0 if $y = 0$ for all x; this gives rises to the only constant solution of the differential equation, which in turn corresponds to $A = 0$ in (2). So the general solution (i.e. the set of all solutions) of (1) is $\boxed{y(x) = C\sqrt{x^2+1}}$ where C is now any constant.

Note: It is good to check that the function $y = y(x)$ is indeed a solution of (1) by computing y', $(x^2+1)y'$, and xy and comparing the last two.

Section 7.3, Problem 10 (15pts)

Find the solution to the initial-value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos x}{1+y^2}, \qquad y(0) = 1. \tag{3}$$

First find the general solution of the differential equation. This is a separable equation, so we can move everything involving y to LHS and everything involving x to RHS, and then integrate:

$$\begin{aligned}\frac{1+y^2}{y}\mathrm{d}y = \cos x\,\mathrm{d}x \quad &\Longleftrightarrow \quad \int \big(y^{-1}+y\big)\mathrm{d}y = \int \cos x\,\mathrm{d}x \\ &\Longleftrightarrow \quad \ln|y| + \frac{1}{2}y^2 = \sin x + C,\end{aligned} \tag{4}$$

where C is any constant. Exponentiating both sides of (4) gives

$$\begin{aligned}\mathrm{e}^{\ln|y|+y^2/2} = \mathrm{e}^{\sin x + C} \quad \Longleftrightarrow \quad \mathrm{e}^{\ln|y|}\mathrm{e}^{y^2/2} = \mathrm{e}^{\sin x}\mathrm{e}^{C} \quad &\Longleftrightarrow \quad |y|\mathrm{e}^{y^2} = A\mathrm{e}^{\sin x} \\ &\Longleftrightarrow \quad y\mathrm{e}^{y^2} = \pm A\mathrm{e}^{\sin x};\end{aligned} \tag{5}$$

since C is any constant, $A = e^C$ is any *positive* constant. However, we divided both sides of (3) by y. This is 0 if $y = 0$ for all x; this gives rises to the only constant solution of the differential equation, which in turn corresponds to $A = 0$ in (5). So the general solution (i.e. the set of all solutions) of (3) is

$$y(x)e^{y(x)^2/2} = C\mathrm{e}^{\sin x} \tag{6}$$

where C is now any constant; this equation defines $y = y(x)$ implicitly as a function of x (for each constant C and x (in some open interval), (6) has a solution $y_C(x)$, possibly more than one).

We need to find C so that the function $y = y(x)$ defined by (6) satisfies the initial condition $y(0) = 1$ in (3). For this, plug in $x = 0$ and $y = 1$ in (6):

$$1 \cdot \mathrm{e}^{1/2} = C\mathrm{e}^{\sin 0} = C\mathrm{e}^0 = C.$$

So $C = \mathrm{e}^{1/2}$ and the solution to the initial-value problem is implicitly defined by

$\boxed{y(x)\mathrm{e}^{y(x)^2} = \mathrm{e}^{1/2}\mathrm{e}^{\sin x} = \sqrt{\mathrm{e}}\mathrm{e}^{\sin x} = \mathrm{e}^{\sin x + 1/2}}$ (either of these three forms is fine)

Note 1: It is good to check that the function $y = y(x)$ above actually solves (3): so compute y' to compare it with $y \cos x/(1 + y^2)$ and check that $y(0) = 1$ as required by the initial condition in (3). Since the latter is easier, it should probably be done first.

Note 2: Another way to find the solution to (3) is to find C in the last expression in (4) by plugging in the initial condition $(x, y) = (0, 1)$:

$$\ln 1 + \frac{1}{2}1^2 = \sin 0 + C \qquad \Longrightarrow \qquad 0 + \frac{1}{2} = 0 + C.$$

So, $C = 1/2$ and the solution $y = y(x)$ to (3) satisfies

$$\ln |y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}.$$

Exponentiating this gives

$$|y|\mathrm{e}^{y^2/2} = \mathrm{e}^{\sin x + 1/2} = \mathrm{e}^{1/2}\mathrm{e}^{\sin x} \qquad \Longrightarrow \qquad y(x)\mathrm{e}^{y(x)^2/2} = \pm\sqrt{\mathrm{e}}\mathrm{e}^{\sin x}.$$

Since $y(0) = 1 > 0$, the sign $\pm$ must be $+$ for the solution to (3). Finding the correct constant as early as possible is generally easier, though less systematic than the first approach.

Note 3: It is not possible to solve the equation $y\mathrm{e}^{y^2/2} = \mathrm{e}^{\sin x + 1/2}$ for y as an explicit function $y = y(x)$. However, it is possible to sketch the graph of $y = y(x)$ anyway; see Figure 1. The *most important* feature of this graph is that it passes through $(0, 1)$ as specified by the initial condition in (3). Since $\sin(x + 2\pi) = \sin x$ for all x,

$$\mathrm{e}^{\sin(x+2\pi)+1/2} = \mathrm{e}^{\sin x + 1/2} \qquad \Longrightarrow \qquad y(x + 2\pi) = y(x);$$

so $y(x)$ is periodic with period 2π. Furthermore, $y(\pi/2 - x) = y(\pi/2 + x)$ since the same is the case for $\sin(x)$, so $y(x)$ is symmetric about $\pi/2 + 2\pi k$, as well as about $3\pi/2 + 2\pi k$. The maximum and minimum values, $y_{\max}$ and $y_{\min}$, of $y = y(x)$ are reached at $x = \pi/2 + 2\pi k$ and $x = 3\pi/2 + 2\pi k$, respectively, because this is the case for $\mathrm{e}^{\sin x+1/2}$ *and* $y\mathrm{e}^{y^2}$ is an increasing function of x for $y > 0$. The two extreme values are of y are defined by

$$y_{\max}\mathrm{e}^{y_{\max}^2/2} = \mathrm{e}^{3/2}, \qquad y_{\min}\mathrm{e}^{y_{\min}^2/2} = \mathrm{e}^{-1/2}.$$

In particular, $y_{\max} \in (3/2, 2)$ and $y_{\min}$ is slightly above $1/2$.

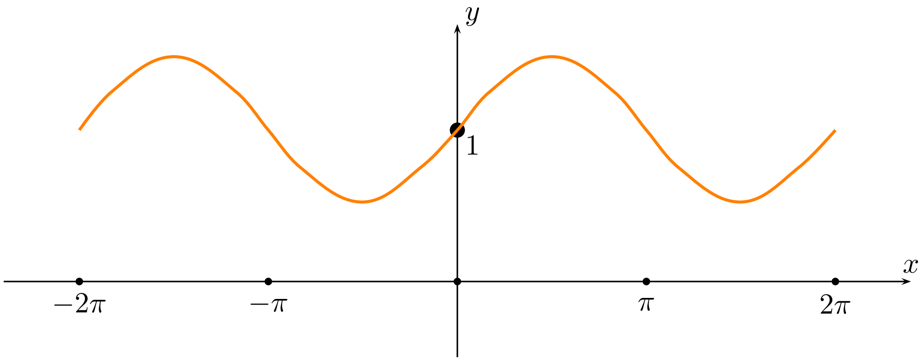

Figure 1: Graph of the solution to the initial-value problem in Section 7.3, Problem 18

Section 7.3, Problem 16 (15pts)

Find an equation for the curve passing through (1,1) and whose slope at (x, y) is y^2/x^3.

The slope of the graph of $y = y(x)$ at $(x, y(x))$ is $y'(x)$. So we need to solve the initial-value problem

$$y' = \frac{y^2}{x^3}, \qquad y(1) = 1. \tag{7}$$

First find the general solution of the differential equation. This is a separable equation, so after writing $y' = \mathrm{d}y/\mathrm{d}x$, we can move everything involving y to LHS and everything involving x to RHS, and then integrate:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x^3} \quad &\Longleftrightarrow \quad \frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}x}{x^3} \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}y}{y^2} = \int \frac{\mathrm{d}x}{x^3} \\ &\Longleftrightarrow \quad -\frac{1}{y} = -\frac{1}{2x^2} + C \quad \Longleftrightarrow \quad y = \frac{2x^2}{1 - 2Cx^2}, \end{aligned} \tag{8}$$

where C is any constant. However, we divided both sides of (7) by y^2. This is 0 if $y = 0$ for all x; this gives rises to the only constant solution of the differential equation. Since the constant solution $y = 0$ does not satisfy the initial condition $y(1) = 1$ in (7), we need to C so that the function y defined by (8) does. For this, plug in $(x, y) = (1, 1)$ into the last equation in (8):

$$1 = \frac{2 \cdot 1^2}{1 - 2C \cdot 1^2} \quad \Longleftrightarrow \quad 1 - 2C = 2 \quad \Longleftrightarrow \quad 2C = -1.$$

So $C = -1/2$, and an equation for the specified curve is $\boxed{y = \frac{2x^2}{1+x^2}}$

Note 1: It is good to check that the function $y = y(x)$ actually solves (7): so compute y' to compare it with y^2/x^3 and check that $y(1) = 1$ as required by the initial condition in (7). Since the latter is easier, it should probably be done first.

Note 2: Another way to find the solution to (7) is to find C in the second-to-last expression in (8) by plugging in the initial condition $(x, y) = (1, 1)$:

$$-\frac{1}{1} = -\frac{1}{2 \cdot 1^2} + C \qquad \Longrightarrow \qquad C = -\frac{1}{2}.$$

So, the solution $y = y(x)$ to (7) satisfies

$$-\frac{1}{y} = -\frac{1}{2 \cdot x^2} - \frac{1}{2} = -\frac{1 + x^2}{2x^2}.$$

This again gives $y(x) = 2x^2/(1 + x^2)$.

Section 7.3, Problem 24 (20pts)

Find the orthogonal trajectories to the family of curves $y^2 = kx^3$. Make a large-size detailed sketch that includes at least 3 members of the original family of curves and at least 3 members of the orthogonal family of curves.

Differentiate the equation $y^2 = kx^3$ with respect to x, using chain rule and remembering that k is a constant:

$$2yy' = k \cdot 3x^2 = 3\frac{y^2}{x}\,, \tag{9}$$

since $k = y^2/x^3$. So our curves have slope $y' = (3/2)y/x$ at (x, y). The slopes of the orthogonal curves are the negative inverses of this; so they satisfy

$$y' = -\frac{2}{3} \cdot \frac{x}{y}. \tag{10}$$

Note that while y in (9) refers to the original curves, y in (10) refers to the orthogonal curves. It is the latter equation we need to solve to find the orthogonal curves (the general solution of the former, $y^2 = kx^3$, is already given).

Equation (10) is separable, so after writing $y' = \mathrm{d}y/\mathrm{d}x$, we can move everything involving y to LHS and everything involving x to RHS and then integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{3} \cdot \frac{x}{y} \quad \Longleftrightarrow \quad y\,\mathrm{d}y = -\frac{2}{3}x\,\mathrm{d}x \quad \Longleftrightarrow \quad \int y\,\mathrm{d}y = -\int \frac{2}{3}x\,\mathrm{d}x$$
$$\Longleftrightarrow \quad \frac{1}{2}y^2 = -\frac{1}{3}x^2 + C \quad \Longleftrightarrow \quad 2x^2 + 3y^2 = A,$$

where $A = 6C$ is any constant. If $A < 0$, the last equation has no (real) solution and so does not correspond to a curve in the real xy-plane. If $A = 0$, the "curve" is just the point $(0, 0)$. If $A > 0$, the corresponding curve in the xy-plane is the ellipse centered at $(0, 0)$, symmetric about the coordinate axes, and with the longer "horizontal radius" equal to $\sqrt{3/2}$ times the short "vertical

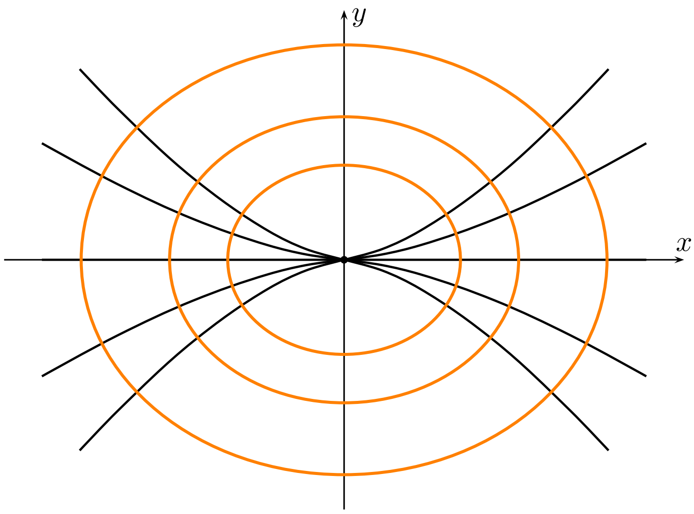

Figure 2: Graph for Section 7.3, Problem 24

radius"; see Figure 2. The original curves are the x-axis (for $k = 0$) and the graphs of the functions $y = C|x|^{3/2}$ with $C = \pm\sqrt{|k|} \neq 0$.

Problem C (40pts)

An example of a function $f = f(y)$ qualitatively matching the graph in Section 7.2, Problem 18 is

$$f(y) = (y^2 - 1)(y^2 - 4).$$

(a; **20pts**) *Find the general solution to the differential equation*

$$y' = (y^2 - 1)(y^2 - 4), \qquad y = y(x).$$

Writing y' as $\mathrm{d}y/\mathrm{d}x$, moving everything involving y to LHS and everything involving x to RHS, and integrating, we obtain

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = (y^2 - 1)(y^2 - 4) \qquad &\Longleftrightarrow \qquad \frac{\mathrm{d}y}{(y^2 - 1)(y^2 - 4)} = \mathrm{d}x \\ &\Longleftrightarrow \qquad \int \frac{\mathrm{d}y}{(y^2 - 1)(y^2 - 4)} = \int 1\mathrm{d}x = x + C. \end{aligned}$$

In order to integrate LHS, we need to use *partial fractions*:

$$\begin{aligned} \frac{1}{(y^2 - 1)(y^2 - 4)} &= \frac{1}{(-1) - (-4)} \left(\frac{1}{y^2 - 4} - \frac{1}{y^2 - 1} \right) \\ &= \frac{1}{3} \cdot \frac{1}{2 - (-2)} \left(\frac{1}{y - 2} - \frac{1}{y + 2} \right) - \frac{1}{3} \cdot \frac{1}{1 - (-1)} \left(\frac{1}{y - 1} - \frac{1}{y + 1} \right). \end{aligned}$$

So the above equality involving integrals is equivalent to

$$\frac{1}{12} \ln|y - 2| - \frac{1}{12} \ln|y + 2| - \frac{1}{6} \ln|y - 1| + \frac{1}{6} \ln|y + 1| = x + C.$$

Multiplying by 12 and using $\ln a + \ln b = \ln(ab)$, we obtain

$$\ln\left(\left|\frac{y-2}{y+2}\right|\left(\frac{y+1}{y-1}\right)^2\right) = 12x + 12C.$$

Exponentiating both sides gives

$$\frac{y-2}{y+2}\left(\frac{y+1}{y-1}\right)^2 = A\mathrm{e}^{12x}\,, \tag{11}$$

where $A = \pm\mathrm{e}^{12C}$ is an arbitrary constant different from 0 (because C is an arbitrary constant). The last equation implicitly defines y as a function of x; according to (b) below, (11) defines $y = y(x)$ uniquely if $A > 0$ and in 3 possible ways if $A < 0$. However, as common with the separation-of-variables method, the last list of solutions does not include the constant ones. These are obtained by solving

$$y' = (y^2-1)(y^2-4) = 0.$$

So the constant solutions are $y = \pm 1, \pm 2$. Two of these are covered by the $A = 0$ case of (11).

$$\boxed{y(x) = -2, 1, \quad \frac{y(x)-2}{y(x)+2}\left(\frac{y(x)+1}{y(x)-1}\right)^2 = A\mathrm{e}^{12x}}$$

(b; **20pts**) *Using the general solution, show that the solutions of the equation qualitatively behave as predicted by the solution to Section 7.2, Problem 18.*

Let $G(y)$ denote RHS of (11). By product rule,

$$\frac{\mathrm{d}G}{\mathrm{d}y} = G(y)\cdot\left(\frac{1}{y-2} - \frac{1}{y+2} + \frac{2}{y+1} - \frac{2}{y-1}\right) = G(y)\cdot\frac{12}{(y^2-1)(y^2-4)}$$
$$= \frac{12}{(y+2)^2(y-1)^2}\left(\frac{y+1}{y-1}\right).$$

Thus, $G(y)$ is strictly increasing in y for $y \in (-\infty,-1)\cup(1,\infty)$ and strictly decreasing for $y \in (-1,1)$ (this does not mean that the solutions $y = y(x)$ do the same though!). We will next consider the solutions of (11), i.e. of

$$G(y) \equiv \frac{y-2}{y+2}\left(\frac{y+1}{y-1}\right)^2 = A\mathrm{e}^{12x}, \qquad A \neq 0, \tag{12}$$

in different ranges of y and compare with the qualitative predictions of the solution to Section 7.2, Problem 18, depicted in Figure 3; see HW2 for comments on this figure.

We begin with the range $y > 2$. As y increases from 2 (not including 2) to ∞, $G(y)$ increases from 0 to 1 (not including either). Thus, if $A > 0$ and $A\mathrm{e}^{12x} < 1$, i.e. $x < -(\ln A)/12$, equation (12) has a unique solution $y_{A;+}(x)$ with $y > 2$; if $x \geq -(\ln A)/12$, this equation has no solutions with $y > 2$. Since $G(y)$ is increasing for $y > 2$ and $A\mathrm{e}^{12x}$ is increasing for $A > 0$,

$$x_1 < x_2 < -(\ln A)/12 \qquad \Longrightarrow \qquad y_{A;+}(x_1) < y_{A;+}(x_2);$$

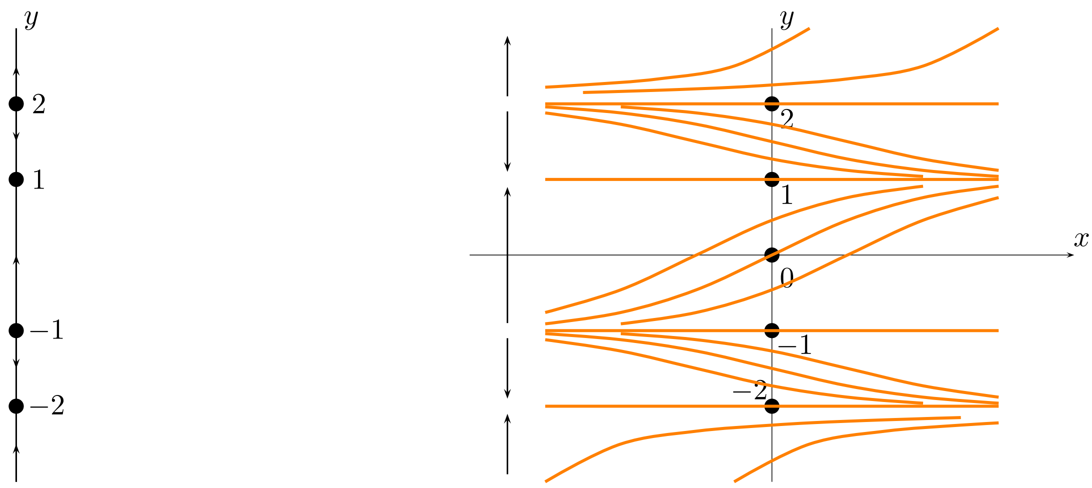

Figure 3: Diagrams of solution curves for Section 7.2, Problem 18

so $y_{A;+}(x)$ is an increasing function. Since $G(y) \longrightarrow 0$ from above as $y \longrightarrow 2$ from above and $Ae^{12x} \longrightarrow 0$ from above for $A > 0$ as $x \longrightarrow -\infty$, $y_{A;+}(x) \longrightarrow 2$ from above as $x \longrightarrow -\infty$. Thus, above the line $y = 2$ solution curves approach $y = 2$ asymptotically as $x \longrightarrow -\infty$ and ascend rapidly to ∞ as x increases; this is as depicted in Figure 3. In fact, these solution curves go off to ∞ in finite time (as $x \longrightarrow -(\ln A)/12$ from below).

The range $y < -2$ is in a sense complementary to the range $y > 2$; the situation is analogous to the parabola $xy = 1$ having two pieces. As y increases from $-\infty$ to -2 (not including -2), $G(y)$ increases from 1 (not including 1) to ∞. Thus, if $A > 0$ and $Ae^{12x} > 1$, i.e. $x > -(\ln A)/12$, equation (12) has a unique solution $y_{A;-}(x)$ with $y < -2$; if $x \le -(\ln A)/12$, this equation has no solutions with $y < -2$. Since $G(y)$ is increasing for $y < -2$ and Ae^{12x} is increasing for $A > 0$,

$$-(\ln A)/12 < x_1 < x_2 \qquad \Longrightarrow \qquad y_{A;-}(x_1) < y_{A;-}(x_2);$$

so $y_{A;-}(x)$ is an increasing function. Since $G(y) \longrightarrow \infty$ as $y \longrightarrow -2$ from below and $Ae^{12x} \longrightarrow \infty$ for $A > 0$ as $x \longrightarrow \infty$, $y_{A;-}(x) \longrightarrow -2$ from below as $x \longrightarrow \infty$. Thus, below the line $y = -2$ solution curves approach $y = -2$ asymptotically as $x \longrightarrow \infty$ and descend rapidly as x decreases; this is as depicted in Figure 3. In fact, these solution curves go off to $-\infty$ in finite time (as $x \longrightarrow -(\ln A)/12$ from above).

In the range $y \in (1, 2)$, $G(y)$ increases from $-\infty$ to 0 (not including 0). Thus, if $A < 0$, equation (12) has a unique solution $y_{A;+}(x)$ with $y \in (1, 2)$. Since $G(y)$ is increasing for $y \in (1, 2)$ and Ae^{12x} is decreasing for $A < 0$,

$$x_1 < x_2 \qquad \Longrightarrow \qquad y_{A;+}(x_1) > y_{A;+}(x_2);$$

so $y_{A;+}(x)$ is a decreasing function. Since $G(y) \longrightarrow 0$ from below as $y \longrightarrow 2$ from below and $Ae^{12x} \longrightarrow 0$ from below for $A < 0$ as $x \longrightarrow -\infty$, $y_{A;+}(x) \longrightarrow 2$ as $x \longrightarrow -\infty$. Since $G(y) \longrightarrow -\infty$ as $y \longrightarrow 1$ and $Ae^{12x} \longrightarrow -\infty$ for $A < 0$ as $x \longrightarrow \infty$, $y_{A;+}(x) \longrightarrow 1$ as $x \longrightarrow \infty$. Thus, in the region between the lines $y = 1$ and $y = 2$, the solution curves descend, asymptotically approaching the line $y = 2$ as $x \longrightarrow -\infty$ and the line $y = 1$ as $x \longrightarrow \infty$; this is again as depicted in Figure 3.

In the range $y \in (-2, -1)$, $G(y)$ also increases from $-\infty$ to 0 (not including 0). So similarly to the previous case, if $A < 0$, equation (12) has a unique solution $y_{A;-}(x)$ with $y \in (-2, -1)$, and $y_{A;-}(x)$

is a decreasing function. Since $G(y) \longrightarrow 0$ from below as $y \longrightarrow -1$ and $Ae^{12x} \longrightarrow 0$ from below for $A < 0$ as $x \longrightarrow -\infty$, $y_{A;-}(x) \longrightarrow -1$ as $x \longrightarrow -\infty$. Since $G(y) \longrightarrow -\infty$ as $y \longrightarrow -2$ from above and $Ae^{12x} \longrightarrow -\infty$ for $A < 0$ as $x \longrightarrow \infty$, $y_{A;-}(x) \longrightarrow -2$ as $x \longrightarrow \infty$. Thus, in the region between the lines $y = -2$ and $y = -1$, the solution curves descend, asymptotically approaching the line $y = -1$ as $x \longrightarrow -\infty$ and the line $y = -2$ as $x \longrightarrow \infty$; this is as depicted in Figure 3.

Finally, in the range $y \in (-1, 1)$, $G(y)$ decreases from 0 (not including 0) to $-\infty$. Thus, if $A < 0$, equation (12) has a unique solution $y_{A;0}(x)$ with $y \in (-1, -1)$. Since $G(y)$ is decreasing for $y \in (-1, 1)$ and Ae^{12x} is decreasing for $A < 0$,

$$x_1 < x_2 \qquad \Longrightarrow \qquad y_{A;0}(x_1) < y_{A;0}(x_2);$$

so $y_{A;0}(x)$ is an increasing function. Since $G(y) \longrightarrow 0$ from below as $y \longrightarrow -1$ and $Ae^{12x} \longrightarrow 0$ from below for $A < 0$ as $x \longrightarrow -\infty$, $y_{A;0}(x) \longrightarrow -1$ as $x \longrightarrow -\infty$. Since $G(y) \longrightarrow -\infty$ as $y \longrightarrow 1$ and $Ae^{12x} \longrightarrow -\infty$ for $A < 0$ as $x \longrightarrow \infty$, $y_{A;0}(x) \longrightarrow 1$ as $x \longrightarrow \infty$. Thus, in the region between the lines $y = -1$ and $y = 1$, the solution curves ascend, asymptotically approaching the line $y = -1$ as $x \longrightarrow -\infty$ and the line $y = 1$ as $x \longrightarrow \infty$; this is as depicted in Figure 3.

In summary, equation (11) has a unique two-piece solution $y_{A;+} = y_{A;+}(x)$ and $y_{A;-} = y_{A;-}(x)$ if $A > 0$; this solution is not defined for and breaks at $x = -(\ln A)/12$. One of the pieces lies in the region $y > 2$, while the other in the region $y < -2$. For $A < 0$, equation (11) has solutions, $y_{A;\pm} = y_{A;\pm}(x)$ and $y_{A;0} = y_{A;0}(x)$; all three of these solutions are defined for all x. There is one solution in each of the three regions $y \in (1, 2)$, $y \in (-2, -1)$, and $y \in (-1, 1)$.

Finally, the sketch is symmetric about the original, i.e. if $y = y(x)$ is a solution of the differential equation in (a), then so is $\tilde{y}(x) = -y(-x)$. This can be seen directly from the differential equation:

$$\begin{aligned}\tilde{y}'(x) &= -y'(-x) \cdot (-1) = \big(y(-x)^2 - 1\big)\big(y(-x)^2 - 4\big) \\ &= \big(\tilde{y}(x)^2 - 1\big)\big(\tilde{y}(x)^2 - 4\big).\end{aligned}$$

This can also be seen from the exact form of the solution (11). If $A > 0$ and $-x > -(\ln(1/A))/12$, so that $y_{1/A;-}(-x)$ is defined, then

$$\begin{aligned} y_{1/A;-}(-x) < -2, \quad & \frac{y_{1/A;-}(-x) - 2}{y_{1/A;-}(-x) + 2}\left(\frac{y_{1/A;-}(-x) + 1}{y_{1/A;-}(-x) - 1}\right)^2 = \frac{1}{A}e^{-12x} \\ \Longrightarrow \qquad -y_{1/A;-}(-x) > 2, \quad & \frac{-y_{1/A;-}(-x) - 2}{-y_{1/A;-}(-x) + 2}\left(\frac{-y_{1/A;-}(-x) + 1}{-y_{1/A;-}(-x) - 1}\right)^2 = Ae^{12x}. \end{aligned}$$

By definition of $y_{A;+}(x)$, the last line above says $y_{A;+}(x) = -y_{1/A;-}(-x)$. The same argument shows that if $A < 0$, then

$$y_{A;+}(x) = -y_{1/A;-}(-x) \qquad \text{and} \qquad y_{A;0}(x) = -y_{1/A;0}(-x),$$

except now we do not have to worry about the domains of the definitions.

Remark: The aims of this problem are to review partial fractions, to indicate the importance of checking for constant solutions of separable equations, and to practice working with implicitly defined functions.