MAT 127: Calculus C, Fall 2009
Homework Assignment 11

Problem Set 11 is due before the beginning of lecture on
Wednesday, 12/09 if enrolled in L01, L02
Thursday, 12/10 if enrolled in L03, L04

Please read Sections 8.7-8.9 thoroughly before starting on the problem set.

Problem Set 11: 8.7 50*,51*,52*,53*,54*; 8.8 2,10,15; 8.9 24**; Problem J (see below)
*also explain why the series converges; **use the first option

Please write your solutions legibly; the graders may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name and lecture number in the upper-right corner of the first page.

Problem J

Power series are typically used to "break" a function into a sequence of numbers (the Taylor coefficients of the function). However, sometimes it is useful to go in the opposite direction, assembling a sequence of numbers into a function.

Let f_n be the n-th Fibonacci number of Example 3c in 8.1,

$$A_n = \sum_{k=1}^{k=n} k = 1+2+\ldots+n, \qquad B_n = \sum_{k=1}^{k=n} k^2 = 1^2+2^2+\ldots+n^2;$$

by definition $f_0 = A_0 = B_0 = 0$.
(a) Use a comparison test and only the definitions of f_n, A_n, and B_n to show that the power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad A(x) = \sum_{n=0}^{\infty} A_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} B_n x^n,$$

have positive radii of convergence (and thus determine well-defined functions near $x=0$).
(b) Using only the definitions of f_n, A_n, and B_n, show that

$$f(x) = x + xf(x) + x^2 f(x), \quad A(x) = xA(x) + \frac{x}{(1-x)^2}, \quad B(x) = xB(x) + \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

(c) Using (b), express f_n, A_n, and B_n explicitly in terms of n.

Hint: You'll need to use identities such as the following, possibly in both directions:

$$\frac{1}{(1-x)^3} = \frac{1}{2}\left(\frac{1}{1-x}\right)'' = \frac{1}{2}\left(\sum_{n=0}^{\infty} x^n\right)'' = \frac{1}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2}.$$

Note: For f_n, you should end up with the formula in Problem F on PS6. There is a much simpler way of finding an explicit formula for A_n; so you can check your answer, but please deduce this formula from (b). The answer for B_n can be confirmed using induction (or google).