MAT 127: Calculus C, Fall 2009
Course Summary III

Extremely Important: sequences vs. series (do not mix them or their convergence/divergence tests up!!!); what it means for a sequence or series to converge or diverge; power series

Very Important: convergence/divergence tests for series; radius and interval of convergence for power series; differentiation, integration, and limits of functions via power series; Taylor series

Important: estimating infinite sums by finite sums; finding radius and interval of convergence of power series; determining Taylor series of functions related to standard ones; applications of power series to computing sums of series

Series (cont'd)

(9) Like the 3 convergence/divergence tests of 8.3 (*Integral Test*, *Comparison Test*, and *Limit Comparison Test*), the Ratio Test is intended for series with positive terms:

> if the sequence $\{a_n\}$ has positive terms and
>
> - $\lim_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges;
> - $\lim_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n} = L > 1$ or $a_{n+1}/a_n \longrightarrow \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

This test says nothing if $a_{n+1}/a_n \longrightarrow 1$. This is not surprising since

$$a_n = \frac{1}{n^p} \qquad \Longrightarrow \qquad \lim_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n} = 1,$$

while the series $\sum a_n$ converges if and only if $p>1$. Thus, the ratio test is *not* suitable for series that involve only powers of n (e.g. n^3), and not something growing faster. On the other hand, it usually works very well with series that involve $n!$ and exponents of n (e.g. 3^n or n^n). For this reason, this is *the* test used to determine the *radius* of convergence of power series in 8.5-8.8 (but other tests are usually required to determine convergence at the end points of the resulting interval). Factors of n do not effect the value of L in the *Ratio Test*; for example, L is the same for all three series

$$\sum_{n=1}^{\infty} \frac{1}{3^n}, \qquad \sum_{n=1}^{\infty} \frac{n}{3^n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n3^n}.$$

In 8.6 and 8.7, this translates into the derivative and anti-derivative of a power series having the same *radius* of convergence as the original series (but the *interval* of convergence may change slightly). Unlike the 3 tests of 8.3, the *Ratio Test* is completely a self-test: you do not have to guess a second sequence $\{b_n\}$, which is required for *Comparison Test* and *Limit Comparison Test*, or a function $f = f(x)$ such that $f(n) = a_n$, which is required for *Integral Test* ("guessing" the function f usually involves replacing n by x if this makes sense (e.g. $x!$ does not); while this is easy, determining whether the resulting integral is finite or not may be less so). The *Ratio Test* is a consequence of the *Comparison Test* applied to a geometric series with $r=(L+1)/2$.

(10) The Alternating Series Test applies to a narrow, but important, set of series with terms of different signs:

> if $\lim_{n\longrightarrow\infty} a_n = 0$, $|a_n| > |a_{n+1}|$, and the signs of a_n alternate ($a_n > 0$ for every n odd and $a_n < 0$ for every n even, or the other way around), then the series $\sum_{n=1}^{\infty} a_n$ converges

The alternating-sign condition is typically exhibited by the presence of $(-1)^n$ or $(-1)^{n-1} = -(-1)^n$; however, make sure to also check the first two conditions before concluding that the series converges. Typical examples are the series like

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\ln n)^2}{n};$$

both converge by the AST. The *Alternating Series Test* is a convergence test only: it states that a series converges if it meets 3 conditions. It can *never* be used to conclude that a series diverges; in this sense, it is the opposite of the most important divergence test, which can *never* be used to conclude that a series converges. If the first condition in the *Alternating Series Test* is not satisfied, the series does indeed diverge, but by the most important divergence test. However, there are lots of series that fail either the second or third condition (or both), but still converge; for example, there are convergent series with only positive terms, that decay to zero, but are not strictly decreasing, e.g.

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n^2}.$$

The *Alternating Series Test* is a consequence of the definition of convergence for series (convergence of the *sequence* of partial sums) and the Monotonic *Sequence* Theorem.

(11) The substance of Absolute Convergence Test is that introducing some minus signs into a convergent series with positive terms does not ruin the convergence:

> if the series $\sum_{n=1}^{\infty} |a_n|$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$

This test is useful when the signs are random, as opposed to strictly alternating as required for the *Alternating Convergence Test*. For example, the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges by the ACT, because the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$

converges since $0 \le |\sin n|/n^2 \le 1/n^2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (this argument uses 3 tests: *Absolute Convergence*, *Comparison*, and *p-Series*; *Limit Comparison Test* is less suitable in this case). The *Alternating Series Test* cannot be applied in this case, because the signs of $\sin n$ do not alternate:

$$\sin 1, \sin 2, \sin 3 > 0, \qquad \sin 4, \sin 5, \sin 6 < 0;$$

while the signs usually come in triples, occasionally there are four consecutive terms with the same sign. While the *Absolute Convergence Test* is less stringent about the alternating sign condition than the *Alternating Series Test*, the former is not a substitute of the latter. While either test can be used to conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges, only the *Alternating Series Test* is applicable to the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ because the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge. Neither of the two tests directly implies that the series $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges[1]. As the *Alternating Series Test*, the *Absolute Convergence Test* is a convergence test only; it can never be used to conclude that a series diverges. The *Absolute Convergence Test* is a consequence of the *Comparison Test* and the addition rule for series.

(12) The general Ratio Test stated in the book extends the above *Ratio Test* for series with positive terms to series with arbitrary nonzero terms (so that a_{n+1}/a_n makes sense):

> if $a_n \neq 0$ for all n ($\geq$ some N), and
>
> - $\lim_{n \longrightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges;
> - $\lim_{n \longrightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $|a_{n+1}/a_n| \longrightarrow \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

The first statement follows from the *Absolute Convergence Test* and the *Ratio Test* for series with positive terms. As with the *Ratio Test* for series with positive terms, the second statement follows from the most important divergence theorem as it implies that $|a_n|$ tends to infinity and thus the sequence $\{a_n\}$ does not converge to 0. Similarly to the *Ratio Test* for series with positive terms, this more general *Ratio Test* says nothing if $|a_{n+1}/a_n| \longrightarrow 1$.

(13) The sum of a convergent series $\sum_{n=1}^{\infty} a_n$ can be estimated by finite sub-sum: the sum

$$s_m = \sum_{n=1}^{n=m} a_n = a_1 + a_2 + \ldots + a_m$$

[1] this series does indeed converge because of a more general version of the *Alternating Series Test*, called *Dirichlet's Test*: if $\{b_n\}$ and $\{s_n\}$ are two sequences such that $\lim_{n \longrightarrow \infty} b_n = 0$, $b_n \geq b_{n+1}$, and there exists $C > 0$ such that $\left| \sum_{n=1}^{n=m} s_n \right| \leq C$ for all m, then the series $\sum_{n=1}^{\infty} s_n b_n$ converges; in the case of the *Alternating Series Test* $s_n = \pm(-1)^n$ is just the sign, and so $C = 1$ works

of the first m terms; this is the m-th partial sum. As $m \longrightarrow \infty$, s_m approaches the sum of the series, so that

$$\sum_{n=m+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n \;-\; s_m \longrightarrow 0.$$

In some cases, the above difference can be estimated:

- if $f = f(x)$ is positive, decreasing, and continuous on $[1, \infty)$ and $\int_1^{\infty} f(x)\mathrm{d}x$ converges, then

$$\boxed{\int_{m+1}^{\infty} f(x)\mathrm{d}x < \sum_{n=m+1}^{\infty} a_n < \int_m^{\infty} f(x)\mathrm{d}x} \qquad (1)$$

  Note that increasing the lower limit (from m to $m+1$ here) makes the integral smaller because $f > 0$. In this case, the finite-sum estimate s_m is an *under*-estimate for the infinite sum because lots of positive terms are dropped from the infinite series.

- if $\lim_{n \longrightarrow \infty} a_n = 0$, $|a_n| > |a_{n+1}|$, and the signs of a_n alternate ($a_n > 0$ for every n odd and $a_n < 0$ for every n even, or the other way around), then

$$\boxed{\left|\sum_{n=m+1}^{\infty} a_n\right| < \left|a_{m+1}\right| \quad \text{and the signs of} \quad \sum_{n=m+1}^{\infty} a_n \quad \text{and} \quad a_{m+1} \quad \text{are the same}} \qquad (2)$$

  In this case, the finite-sum estimate s_m is an *under*-estimate for the infinite sum if $a_m < 0$ and an *over*-estimate if $a_m > 0$ (so determined by the last term used in the estimate).

For example, let's estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to within 1/5. Since $f(x) = 1/x^2 > 0$ is continuous and decreasing on $[1, \infty)$, by (1) we need to find the smallest integer m such that

$$\int_m^{\infty} f(x)\mathrm{d}x = \int_m^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{m} \leq \frac{1}{5}\,.$$

So $m = 5$ and the required finite-sum estimate is

$$\sum_{n=1}^{n=5} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{3600 + 900 + 400 + 225 + 144}{3600} = \boxed{\frac{5269}{3600}}$$

This is an under-estimate for the infinite sum, as only positive terms are dropped off from the latter.

Let's next estimate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ to within 1/5. Since this series is alternating (odd terms > 0, even terms < 0), $1/n \longrightarrow 0$, and $1/(n+1) > 1/n$, by (2) we need to find the smallest integer m such that

$$\left|a_{m+1}\right| = \frac{1}{m+1} \leq \frac{1}{5}\,.$$

So $m = 4$ and the required finite-sum estimate is

$$\sum_{n=1}^{n=4} \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{12 - 6 + 4 - 3}{12} = \boxed{\frac{7}{12}}$$

This is an under-estimate for the infinite sum, as the last term in the estimate is negative.

Remark: The estimates (1) and (2) are closely tied to the *Integral Test* and the *Alternating Series Test* for convergence of series. In principle, there are estimates related to other convergence tests, in particular the *Ratio Test*, but they are not discussed in the textbook.

Power Series

(1) A power series is a function defined by an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \,, \tag{3}$$

where a is some fixed number, typically 0, called the center of the power series (3); plugging in $x=a$ makes all the terms $(x-a)^n$ to be 0^n. So the center of the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$$

is $x=-1$. By convention used in defining power series, $(x-a)^0 = 1$ even if $x=a$. By a similar convention, $0!=1$ so that $\boxed{(n+1)! = n! \cdot (n+1)}$ for all non-negative integers n.

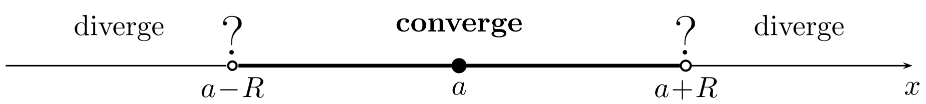
(2) For each x for which the series (3) converges, we obtain a number $f(x)$. In particular,

$$f(a) = c_0 0^0 + c_1 0^1 + c_2 0^2 + \ldots = c_0;$$

so the power series (3) always converges at its center $x=a$. The most fundamental question about a power series is the set of the numbers x for which the power series converges. By the Main Theorem about Power Series, this set can be of one of 3 types, with one types having 4 sub-types:

> The power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges either
>
> (a) for $x=a$ only;
>
> (b) for all x;
>
> (c) for x in one of the intervals $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$ for some $R>0$ and diverges otherwise
>
> (4)

The four possibilities in (c) are illustrated below:



According to this theorem, the set of values of x for which a power series converges cannot be arbitrary; it must be an interval, which is centered at the center of the power series, may consist of a single point, be infinite, or be of finite nonzero length and in the last case can be open, closed, or half-open (so the series can either converge or diverge at each of the two end-points of the interval; this is indicated by the question marks in the sketch). The interval on which a power series converges is its interval of convergence. The number R in (c) is the radius of convergence of the power series; $R=0$ in (a) and $R=\infty$ in (b).

(3) In order to **find the radius of convergence** of a power series as in (3) with $c_n \neq 0$ for all n ($\geq$ some N), use the general *Ratio Test* with $a_n = c_n(x-a)^n \neq 0$ (so we assuming $x \neq a$, since we already know that the power series converges for $x=a$):

$$\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\longrightarrow\infty} \frac{|c_{n+1}| \cdot |x-a|^{n+1}}{|c_n| \cdot |x-a|^{n+1}} = |x-a| \cdot \lim_{n\longrightarrow\infty} \frac{|c_{n+1}|}{|c_n|}. \tag{5}$$

In general, the last limit in (5) does not have to exist. However, in all examples encountered in this class it either exists or $|c_{n+1}|/|c_n| \longrightarrow \infty$. In these cases:

- if $|c_{n+1}|/|c_n| \longrightarrow \infty$, the last number in (5) is ∞; by the general *Ratio Test* the series (3) diverges for every $x \neq a$, and so we are in case (a) of (4) and $R=0$.
- if $|c_{n+1}|/|c_n| \longrightarrow 0$, the last number in (5) is 0; by the general *Ratio Test* the series (3) converges for every $x \neq a$, and so we are in case (b) of (4) and $R=\infty$.
- if $|c_{n+1}|/|c_n| \longrightarrow C \neq 0$, the last number in (5) is $C|x-a|$; by the general *Ratio Test* the series (3) converges if $|x-a| < 1/C$ and diverges $|x-a| > 1/C$. So we are in case (c) of (4) and $R=1/C$.

Once the radius of convergence is found, **find the interval of convergence** of the power series. If $R=0$, then the interval of convergence is just $[a,a]$; if $R=\infty$, then the interval of convergence is $(-\infty,\infty)$. If $R \neq 0, \infty$, it remains to determine whether the power series converges for $x=a-R$ and for $x=a+R$, i.e. you have to determine *separately* whether each of the two power series

$$\sum_{n=0}^{\infty} c_n(-1)^n R^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} c_n R^n$$

converges. You will have to use some convergence/divergence tests for series, but **not** the *Ratio Test* (it would give 1 in the limit and so be inconclusive in these two cases). Once this is done, the interval of convergence will be as in one of the 4 subcases in (c) of (4).

Remark 1: If c_n involves $n!$ in the *numerator* and the remaining terms are powers and exponentials of n, such as $\sqrt{n+1}$ or 2^n (but not n^n), then you'll be in case (a) of (4). If c_n involves $n!$ in the *denominator* and the remaining terms are powers and exponentials of n, such as n^3 or 5^n (but not n^n), then you'll be in case (b) of (4). If c_n involves *only* powers and exponentials of n, such as $n^3/\sqrt{n^2+n}$ or 3^n (but not n^n), then you'll be in case (c) of (4); this is the case when you'll also need to determine whether the series converges or diverges at *each* of the two end-points of the interval of convergence *separately.*

Remark 2: By the above, if $|c_{n+1}|/|c_n| \longrightarrow C$ for some nonnegative number C or for $C=\infty$, then the radius of convergence of the power series is $R=1/C$. In general, $|c_{n+1}|/|c_n|$ may not approach anything, including ∞, because it may keep on jumping. For example, $|c_{n+1}|/|c_n|$ with n odd might approach 0 and $|c_{n+1}|/|c_n|$ with n even might approach ∞; then 0 and ∞ are said to be *limits of subsequences.* There can be lots of such limits of subsequences, but there is always at least one (possibly $\pm\infty$). The largest of such limits is denoted lim sup (and the smallest lim inf). If a sequence converges, lim sup is just its usual limit. If C is lim sup of the sequence $|c_{n+1}|/|c_n|$, then the radius of convergence of the power series (3) is still $R=1/C$. You can learn more about lim sup in MAT 320.

(4) If the radius of convergence of a power series is 0, the power series is rather useless. However, if its radius of convergence R is positive (possibly ∞), it defines a *smooth* function $f(x)$ on $(a–R, a+R)$. This function can be differentiated and integrated by differentiating and integrating the power series term by term (like a polynomial):

If the radius of convergence R of the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is positive (possibly ∞),

the function $\quad f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad$ is smooth on the open interval $(a-R, a+R)$,

- $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$, the radius of convergence of this power series is still R, and if $R \neq \infty$ and the series for f *diverges* for $x=a\pm R$, so does the series for f';
- $\int f(x)\mathrm{d}x = C+\sum_{n=0}^{\infty} \frac{c_n}{n+1}(x–a)^{n+1} = C+\sum_{n=1}^{\infty} \frac{c_{n-1}}{n-1}(x–a)^n$, the radius of convergence of this power series is still R, and if $R\neq\infty$ and the series for f *converges* for $x=a\pm R$, so does the series for $\int f(x)\mathrm{d}x$.

(6)

So the radius of convergence of a power series does not change under differentiation and integration, but the interval of converge may change if $R \neq \infty$: differentiation may remove one or both of the endpoints from the interval of convergence, while integration may add them to the interval convergence. For example, the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \tag{7}$$

converges if and only if $|x|<1$ and for each x such that $|x|<1$ it converges to $1/(1–x)$. So the radius of convergence of the series (7) is 1, its interval of convergence is $(-1, 1)$, and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \qquad \text{if} \quad |x| < 1. \tag{8}$$

Differentiating both sides of (8) with respect to x gives

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \dots \qquad \text{if} \quad |x| < 1. \tag{9}$$

The radius of convergence of this power series is still 1, while the interval of convergence is still $(-1, 1)$ because there are no ends to potentially drop off from the interval of convergence of the power series (8). Integrating both sides of (8) from $x=0$ (this makes $C=0$ in (6)) gives

$$-\ln(1-x) = \int_0^x \frac{1}{(1-u)}\mathrm{d}u = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots. \tag{10}$$

The radius of convergence of this power series is still 1, but the interval of convergence may have increased by either or both of the two endpoints; this has to be checked separately. Setting $x=1$ in the series in (10) gives $\sum_{n=1}^{\infty}\frac{1}{n}$; this series diverges by the *p-Series Test.* Setting $x=-1$ in the series in (10) gives $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$; this series converges by the *Alternating Series Test.* Thus, the interval of convergence of the power series in (10) is $[-1,1)$ and

$$-\ln(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots \qquad \text{if} \quad -1\le x<1. \tag{11}$$

Taking $x=1/2$ and $x=-1$ in (11) gives

$$-\ln(1/2)=\sum_{n=1}^{\infty}\frac{(1/2)^n}{n}=\sum_{n=1}^{\infty}\frac{1}{n2^n}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\ldots\,,$$

$$-\ln(2)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=-\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\ldots\,,$$

Since $-\ln(1/2)=\ln(2)$, we find that

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n2^n}=\ln 2=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}} \tag{12}$$

Remark 1: You do not need to memorize the two formulas in (12), but you need to understand and be able to apply the principles involved in obtaining them; in particular, you have to be able to find sums of analogous infinite series. You have to remember the formulas for differentiating and integrating power series given in (6); remembering the first of the two formulas for each should suffice and should be fairly easy, since these are just differentiation and integration of (infinite) polynomials. If you asked to find the sum of an infinite series as in (12), you need to be able to see that it is obtained from some *power series* by replacing x with a specific value in the range of the convergence of x. You should then be able to recognize the power series and know what function it sums up to, at least after dropping same factors of n from all terms. By (6), extra factors of n correspond to differentiation or integration of the power series you recognize (but be careful to check that the exponents of x are correct and not shifted by a fixed number; if they are shifted, just take a power of x outside of the summation). You can then determine the function to which the original power series corresponds and sum up the starting infinite series by evaluating this function at the appropriate value of x.

Remark 2: the statements of (6) regarding the radii of convergence follow from Remark 2 in (3) above; so you can actually verify them assuming $|c_{n+1}|/|c_n|\longrightarrow C$ for some $C\ge 0$ (possibly ∞).

(5) Limits of functions defined via power series can be easily computed, as long as the limit is taken at the center of the power series. This generally involves writing out the first few terms of the power series. For example, the function

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$$

is defined whenever $-1\le x\le 1$ and thus for all x close to 0; so it makes sense to ask about limits of related functions at $x\!=\!0$. In particular,

$$\lim_{x\longrightarrow 0}\frac{f(x)-x-\frac{1}{2}x^2}{x^3}=\lim_{x\longrightarrow 0}\frac{\left(\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots\right)-x-\frac{1}{2}x^2}{x^3}=\lim_{x\longrightarrow 0}\frac{\left(\frac{x^3}{3}+\frac{x^4}{4}+\ldots\right)}{x^3}$$
$$=\lim_{x\longrightarrow 0}\left(\frac{1}{3}+\frac{1}{4}x+\ldots\right)=\frac{1}{3};$$

on the second line ... denotes terms involving x and higher powers of x, all of which approach 0 as $x\longrightarrow 0$. You can compute this limit using l'Hospital's rule as well, but it would have to be applied *3 times*, re-checking the required assumptions each time (in this case, this would mean checking that the numerator and denominator both approach 0).

(6) Two power series with the same center, say 0,

$$f(x)=\sum_{n=0}^{\infty}f_nx^n \qquad \text{and} \qquad g(x)=\sum_{n=0}^{\infty}g_nx^n$$

can be multiplied together by treating them as infinite polynomials and collecting coefficients of each power of $(x-a)$:

$$\begin{aligned}f(x)g(x)&=\Big(\sum_{n=0}^{\infty}f_nx^n\Big)\Big(\sum_{n=0}^{\infty}g_nx^n\Big)\\&=\Big(f_0+f_1x+f_2x^2+f_3x^3+\ldots\Big)\Big(g_0+g_1x+g_2x^2+g_3x^3+\ldots\Big)\\&=f_0g_0+(f_0g_1+f_1g_0)x+(f_0g_2+f_1g_1+f_2g_0)x^2+(f_0g_3+f_1g_2+f_2g_1+f_3g_0)x^3+\ldots\\&=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{k=n}f_kg_{n-k}\Big)x^n.\end{aligned}$$

So the coefficient of x^n in the product is the sum $n\!+\!1$ terms, each of which is a product of a term from the f-series and a term from the g-series. If the f and g-series converge for $|x|<R$, then so does the fg-series. For example,

$$\begin{aligned}\frac{1}{(1-x)^2}=\frac{1}{1-x}\cdot\frac{1}{1-x}&=\Big(\sum_{n=0}^{\infty}x^n\Big)\Big(\sum_{n=0}^{\infty}x^n\Big)\\&=1\cdot1+(1\cdot1+1\cdot1)x+(1\cdot1+1\cdot1+1\cdot1)x^2+\ldots=\sum_{n=1}^{\infty}nx^{n-1};\end{aligned}$$

this agrees with (9), as well as with the $k\!=\!-2$ case of (19) below. A more interesting example is

$$\begin{aligned}\frac{1}{x^2-3x+2}&=\frac{1}{(x-1)(x-2)}=\frac{1}{1-x}\cdot\frac{1/2}{1-x/2}\\&=\frac{1}{2}\Big(\sum_{n=0}^{\infty}x^n\Big)\Big(\sum_{n=0}^{\infty}\Big(\frac{x}{2}\Big)^n\Big)=\frac{1}{2}\Big(\sum_{n=0}^{\infty}x^n\Big)\Big(\sum_{n=0}^{\infty}\frac{x^n}{2^n}\Big)\\&=\frac{1}{2}\Big(1\cdot1+\Big(1\cdot\frac{1}{2}+1\cdot1\Big)x+\Big(1\cdot\frac{1}{4}+1\cdot\frac{1}{2}+1\cdot1\Big)x^2+\ldots\Big)\\&=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1^{n+1}-(1/2)^{n+1}}{1-1/2}x^n=\sum_{n=0}^{\infty}\Big(1-\frac{1}{2^{n+1}}\Big)x^n\end{aligned}$$

The second-to-last equality above is the $(a,b)=(1,1/2)$ case of

$$\begin{aligned} a^{n+1}-b^{n+1} &= (a-b)\big(a^n b^0 + a^{n-1}b^1 + \ldots + a^1 b^{n-1} + a^0 b^n\big) \\ &= (a-b)\big(b^n + ab^{n-1} + \ldots + a^{n-1}b + a^n\big); \end{aligned}$$

this formula was used to sum up geometric series. Another way to expand $1/(x^2-3x+2)$ around $x=0$ is to use *partial fractions* and addition of series, instead of multiplication (addition is much simpler):

$$\begin{aligned} \frac{1}{x^2-3x+2} &= \frac{1}{(x-1)(x-2)} = \frac{1}{-2-(-1)}\left(\frac{1}{x-1}-\frac{1}{x-2}\right) = -\frac{1}{x-1}+\frac{1}{x-2} \\ &= \frac{1}{1-x}-\frac{1/2}{1-x/2} = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)x^n \end{aligned}$$

The interval of convergence of this power series can easily be seen to be $(-1,1)$.

(7) There is *at most* one way to expand a function into a power series centered at a given point:

$$\boxed{\begin{array}{l} \text{If } f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \text{ on } (a-R,a+R) \text{ for some } c_0,c_1,\ldots \text{ and } R>0, \text{ then} \\ \bullet\ f \text{ is a smooth function on } (a-R,a+R); \\ \bullet\ c_n = \dfrac{f^{\langle n\rangle}(a)}{n!}, \text{ where } f^{\langle n\rangle}=f^{\langle n\rangle}(x) \text{ is the } n\text{-th derivative of } f,\ f^{\langle 0\rangle}=f,\ 0!=1. \end{array}} \tag{13}$$

A power series expansion of f around $x=a$, if it exists, is called the Taylor series expansion of f around $x=a$. By the first statement in (13), the function $f(x)=|x|$ does not admit a Taylor series expansion around $x=a$, though it does admit such an expansion around any $x=a\neq 0$:

$$|x| = \begin{cases} a+1\cdot(x-a) \ \text{ on } (0,\infty), & \text{if } a>0; \\ -a-1\cdot(x-a) \ \text{ on } (-\infty,0), & \text{if } a<0. \end{cases}$$

The second statement in (13) provides a method of determining the Taylor coefficients c_n of f at $x=a$. However, this method is practical only if *all* derivatives of f can be computed; this can be done in some cases, including

- $f(x)=p(x)$ is a polynomial of degree d; then

$$p(x) = p(a) + p'(x)(x-a) + \frac{p''(a)}{2!}(x-a)^2 + \ldots + \frac{p^{\langle d\rangle}(a)}{d!}(x-a)^d\,, \tag{14}$$

  because $p^{\langle n\rangle}(x)=0$ if $n>d$; furthermore, $p^{\langle d\rangle}(a)/d!=1$. The "series" on the right-hand side of (14) converges for all x because it is a finite sum. The equality of the left and right expressions in (14) also holds for all x, because of *Taylor's Inequality* below (not because the right expression is a finite sum).

- $f(x)=1/(1-x)$: in this case $f^{\langle n\rangle}(x)=n!/(1-x)^{n+1}$ as can be seen by induction, and so

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty}\frac{n!/1^{n+1}}{n!}x^n = \sum_{n=0}^{\infty} x^n \qquad \text{if} \quad |x|<1. \tag{15}$$

  The series on the right-hand side is a geometric series with $r=x$. It converges if and only if $|x|<1$; if so, it converges to $1/(1-x)$ as stated in (15).

- $f(x)=\mathrm{e}^x$: in this case $f^{\langle n\rangle}(x)=\mathrm{e}^x$ for all n and

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{\mathrm{e}^0}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,. \tag{16}$$

The series on the right-hand side converges for all x by the Ratio Test. The equality of the left and right expressions in (16) also holds for all x, because of *Taylor's Inequality* below (not because the series converges for all x).

- $f(x)=\cos x$: in this case

$$f^{\langle 4n\rangle}(x) = \cos x\,, \quad f^{\langle 4n+1\rangle}(x) = -\sin x\,, \quad f^{\langle 4n+2\rangle}(x) = -\cos x\,, \quad f^{\langle 4n+3\rangle}(x) = \sin x$$

for all n, as can be seen by induction. Since $\cos 0 = 1$ and $\sin 0 = 0$,

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cos 0}{(2n)!}x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\,. \tag{17}$$

The series on the right-hand side converges for all x by the Ratio Test. The equality of the left and right expressions in (17) also holds for all x, because of *Taylor's Inequality* below (not because the series converges for all x).

- $f(x)=\sin x$: in this case

$$f^{\langle 4n\rangle}(x) = \sin x\,, \quad f^{\langle 4n+1\rangle}(x) = \cos x\,, \quad f^{\langle 4n+2\rangle}(x) = -\sin x\,, \quad f^{\langle 4n+3\rangle}(x) = -\cos x$$

for all n, as can be seen by induction. Since $\cos 0 = 1$ and $\sin 0 = 0$,

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cos 0}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{18}$$

The series on the right-hand side converges for all x by the Ratio Test. The equality of the left and right expressions in (18) also holds for all x, because of *Taylor's Inequality* below (not because the series converges for all x).

- $f(x)=(1+x)^k$, k is any real number and $|x|<1$ (so that $(1+x)^k$ is defined even if k is not integer): in this case

$$f^{\langle n\rangle}(x) = k(k-1)(k-2)\dots(k-n+1)(1+x)^{k-n}$$

for all n, as can be seen by induction. Thus,

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \binom{k}{n}(1+0)^{k-n}x^n = \sum_{n=0}^{\infty} \binom{k}{n}x^n \qquad \text{if} \quad |x|<1, \tag{19}$$

$$\text{where} \qquad \binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!}.$$

The radius of convergence of the series on the right-hand side is 1 by the Ratio Test, unless k is a non-negative integer (in which case RHS of (19) is a finite sum and so converges for all x). The equality of the left and right expressions in (19) also holds for $|x|<1$, because of 8.8 15 (not because the series converges if $|x|<1$). The identity (19) is known as binomial series. The geometric series (15) is a special case of (19), with $k=-1$ and x replaced by $-x$. If k is a non-negative integer, so that $(1+x)^k$ is a polynomial, for all $n>k$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\ldots 0\ldots(k-n+1)}{n!} = 0$$

and so the series (19) has only finitely many terms, and the identity holds for all x.

In many cases, it is not practical to compute all derivatives of a function and so it may not be possible to use the formula in (13) to compute the Taylor coefficients directly. However, it may be possible to obtain the Taylor expansion for a given function by using one of the "standard" series (15)-(18). For example,

$$x^5 e^{-3x^2} = x^5 \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} = x^5 \sum_{n=0}^{\infty} \frac{(-3)^n (x^2)^n}{n!} = x^5 \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+5}}{n!} ;$$

since the Taylor series for e^x converges for all x, so does the above Taylor series for $x^5 e^{-3x^2}$. Similarly,

$$\frac{x^5}{1+3x^2} = \frac{x^5}{1-(-3x^2)} = x^5 \sum_{n=0}^{\infty} (-3x^2)^n = x^5 \sum_{n=0}^{\infty} (-3)^n x^{2n} = \sum_{n=0}^{\infty} (-3)^n x^{2n+5} .$$

Since the Taylor series for $1/(1-x)$ converges if $|x|<1$, the above Taylor series converges if $|-3x|<1$ (whatever is used for x in the power series also has to be used in the bound for convergence); so it converges if $|x|<1/\sqrt{3}$.

Remark 1: When you use a Taylor series for one function to get a Taylor series expansion for another function, make sure your final answer is a power series in x (or $(x–a)$ if the center $a\neq 0$), not a power series in, say, $-3x^2$ or x^2, and not a product of a power series with, say, x^5 (see the two examples above). While there are many different ways to describe a function, there is at most **one way** to write it a power series.

Remark 2: You should remember the formula for the Taylor coefficients c_n in (13) or equivalently the general Taylor expansion formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{\langle n \rangle}(a)}{n!} (x-a)^n ;$$

this formula is often used with $a=0$. On the other hand, the four formulas (15)-(18) will be provided for the exam (see the last page of *Final Exam Information*). You can take these four power series expansions, along with their intervals of convergence, as given and use them as appropriate. They may be helpful in obtaining other Taylor series, computing limits, and computing sums of infinite series. For full credit, you must derive any other power series formula you use on the exam, either directly from the Taylor coefficient formula in (13) or from one of the four given Taylor series. In particular, you may need to derive a formula like (14) for a specific polynomial $p(x)$ around specified center $x=a$; see 8.7 9,10 for examples. You should *not* memorize the binomial formula (19); if anything related, with a specific k, appears on the exam, you should not quote the binomial formula anyway.

Remark 3:[2] The key to determining whether a given function admits Taylor series expansion around a given point is Taylor's Inequality:

$$\left|f(x)-\sum_{n=0}^{n=m}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n\right|\le C_{m+1}(f;R)\frac{R^{m+1}}{(m+1)!}\qquad \text{if } a-R\le x\le a+R, \tag{20}$$

where $C_{m+1}(f;R)$ is the maximum value of $|f^{\langle m+1\rangle}(x)|$ with x in $[a-R,a+R]$. This inequality is obtained as follows. Let

$$R_{m+1}(x)=f(x)-\sum_{n=0}^{n=m}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n.$$

Then,

$$R_{m+1}(a)=0,\quad R'_{m+1}(a)=0,\quad \ldots,\quad R^{\langle m\rangle}_{m+1}(a)=0.$$

Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
R^{\langle m\rangle}_{m+1}(x)&=R^{\langle m\rangle}_{m+1}(a)+\int_a^x R^{\langle m+1\rangle}_{m+1}(u)\mathrm{d}u=\int_a^x R^{\langle m+1\rangle}_{m+1}(u)\mathrm{d}u\\
R^{\langle m-1\rangle}_{m+1}(x)&=R^{\langle m-1\rangle}_{m+1}(a)+\int_a^x R^{\langle m\rangle}_{m+1}(u)\mathrm{d}u=\int_a^x R^{\langle m\rangle}_{m+1}(u)\mathrm{d}u\\
&\vdots\\
R_{m+1}(x)=R^{\langle 0\rangle}_{m+1}(x)&=R^{\langle 0\rangle}_{m+1}(a)+\int_a^x R^{\langle 1\rangle}_{m+1}(u)\mathrm{d}u=\int_a^x R^{\langle 1\rangle}_{m+1}(u)\mathrm{d}u
\end{aligned}$$

Thus, for all x in $[a,a+R]$:

$$\begin{aligned}
\big|R^{\langle m\rangle}_{m+1}(x)\big|&\le\int_a^x|R^{\langle m+1\rangle}_{m+1}(u)|\mathrm{d}u\le\int_a^x C_{m+1}(f;R)\mathrm{d}u\le C_{m+1}(f;R)|x-a|\\
\big|R^{\langle m-1\rangle}_{m+1}(x)\big|&\le\int_a^x|R^{\langle m\rangle}_{m+1}(u)|\mathrm{d}u\le\int_a^x C_{m+1}(f;R)|x-a|\mathrm{d}u\le\frac{C_{m+1}(f;R)}{2!}|x-a|^2\\
&\vdots\\
\big|R^{\langle 0\rangle}_{m+1}(x)\big|&\le\int_a^x|R^{\langle 1\rangle}_{m+1}(u)|\mathrm{d}u\le\int_a^x\frac{C_{m+1}(f;R)}{m!}|x-a|^m\mathrm{d}u\le\frac{C_{m+1}(f;R)}{(m+1)!}|x-a|^{m+1}
\end{aligned}$$

The same estimates holds if x lies in $[a-R,a]$. This confirms (20). By (20), if

$$\lim_{m\longrightarrow\infty}C_n(f;R)\frac{R^n}{n!}=0,$$

then

$$f(x)=\lim_{m\longrightarrow\infty}\sum_{n=0}^{n=m}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(a)}{n!}(x-a)^n\,,$$

and so the Taylor series for f does indeed equal to f on $[a-R,a+R]$. In particular, this is the case if

- $f(x)=p(x)$ is a polynomial of degree d: since $p^{\langle n\rangle}(x)=0$ if $n>d$, $C_n(f;R)=0$;
- $f(x)=\mathrm{e}^x$, $a=0$: since $f^{\langle n\rangle}(x)=\mathrm{e}^x$ for all n, $C_n(f;R)=\mathrm{e}^R$;

[2] you can skip this remark in studying for the final exam

- $f(x)=\cos x$ or $f(x)=\sin x$: since $f^{\langle n\rangle}(x)$ is $\pm\cos x$ or $\pm\sin x$, $C_n(f;R)=1$ if $R\geq\pi$.

(8) Power/Taylor series can be used to compute sums of some convergent infinite series, $\sum\limits_{n=0}^{\infty} a_n$, and even check convergence (in some cases only). Begin by writing the infinite series as evaluation of some power series at some point:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n(x-a)^n\Big|_{x=b};$$

so you need to guess an appropriate sequence $\{c_n\}$ as well as the center of the series a and the evaluation point b, but in some cases they may be evident. For example,

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}=\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{1}{2}\Big)^n=\sum_{n=1}^{\infty}\frac{1}{n}x^n\Big|_{x=\frac{1}{2}}, \qquad \sum_{n=1}^{\infty}\frac{n}{2^n}=\sum_{n=1}^{\infty}n\Big(\frac{1}{2}\Big)^n=\sum_{n=1}^{\infty}nx^n\Big|_{x=\frac{1}{2}}.$$

You next need to find a simple formula for the function $g(x)=\sum_{n=0}^{\infty}c_n(x-a)^n\Big|_{x=b}$. It may not be one of the standard Taylor series, but may become such after dropping a fraction involving powers of n. For example,

$$\sum_{n=1}^{\infty}\frac{1}{n}x^n, \sum_{n=1}^{\infty}nx^n \quad\longrightarrow\quad \sum_{n=0}^{\infty}x^n=\frac{1}{1-x}.$$

In light of (6), the function $g=g(x)$ can then be reconstructed from the function $f=f(x)$ through differentiation and/or integration and possible multiplication by a power of x after each step to account for differences in the exponent if any; in the case of integration, the constant C has to be chosen appropriately as well. For example,

$$\sum_{n=1}^{\infty}\frac{1}{n}x^n+C=\int\Big(\sum_{n=1}^{\infty}x^{n-1}\Big)\mathrm{d}x=\int\frac{\mathrm{d}x}{1-x}=-\ln(1-x)+C' \quad\Longrightarrow\quad \sum_{n=1}^{\infty}\frac{1}{n}x^n=-\ln(1-x);$$

$$\sum_{n=1}^{\infty}nx^n=x\sum_{n=1}^{\infty}nx^{n-1}=x\Big(\sum_{n=0}^{\infty}x^n\Big)'=x\Big(\frac{1}{1-x}\Big)'=\frac{x}{(1-x)^2};$$

the last equality on the first line is obtained by setting $x=0$. The interval of convergence for the g-series can be determined from the f-series. For example, the radii of convergence of both series

$$\sum_{n=1}^{\infty}\frac{1}{n}x^n=-\ln(1-x), \qquad \sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2}, \tag{21}$$

are 1, since this is the radius of convergence of the geometric series $\sum\limits_{n=1}^{\infty}x^n$ and the *radius* of convergence of a power series does not change under differentiation or integration. Since the interval of convergence of $\sum\limits_{n=1}^{\infty}x^n$ is $(-1,1)$, this is also the interval of convergence of the power series $\sum\limits_{n=1}^{\infty}nx^n$ since differentiation can only remove (but not necessarily) the end-points from the interval of convergence (and there are not any end-points to remove in this case). On the other hand, integration can only add in the end-points; since $\sum\limits_{n=1}^{\infty}\frac{1}{n}x^n$ converges for $x=-1$ and diverges at $x=1$, the interval of convergence of this power series is $[-1,1)$. Once it is established that the required evaluation point b lies inside of the interval of convergence of the g-series, we obtain

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n (x-a)^n \Big|_{x=b} = g(b).$$

For example, since $1/2$ lies in the intervals of convergence of the two power series in (21),

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} x^n \Big|_{x=\frac{1}{2}} = -\ln(1-x)\big|_{x=\frac{1}{2}} = -\ln(1/2) = \ln(2),$$
$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n x^n \Big|_{x=\frac{1}{2}} = \frac{x}{(1-x)^2}\Big|_{x=\frac{1}{2}} = \frac{1/2}{1/4} = 2;$$

make sure to simplify the final answer as much as possible.

Remark: If the g-series is already a standard series, you do not need to do any differentiation or integration; just plug in the evaluation point to get the sum, as long as the evaluation point lies in the interval of convergence of the power series. In all cases, if you actually know that the infinite series converges, the evaluation point automatically lies in the interval of convergence. If you are asked to justify that the series converges, you either need to use one of the convergence/divergence tests for series or show that the evaluation point lies inside of the interval of convergence. In order to do so, it is often sufficient to determine just the *radius* of convergence: if the distance from the center to the evaluation point is (strictly) less than the radius of convergence, than the evaluation point lies in the interval of convergence. This is the case in the two examples above, since the distance from 0 to 1/2 is less than 1.

Convergence/Divergence Tests for Sequences and Series (recap)

The two most important things regarding Chapter 8 are

- distinguishing between sequences and series and their convergence/divergence tests;
- realizing that the convergence/divergence issue concerns what happens with "the infinite tail". Thus, dropping the first 159 terms of a sequence or series will not change its convergence/divergence property. If a series does converge, dropping the first 159 terms will however change the sum of the infinite series, precisely by the sum of the first 159 terms.

Confusion about these two points, especially the first one, was the primary reason for the low scores on the second midterm; not knowing which convergence/divergence test was rather secondary.

Whether a sequence/series converges or diverges depends primarily the dominant terms and the presence of any sign-alternating or oscillatory behavior, such as $(-1)^n$ or $\sin n$; factors like $\sin(1/n)$ and $\cos(1/n)$ are not oscillatory, since they approach 0 and 1, respectively, as $n \longrightarrow 0$. It is generally helpful to try to isolate the dominant terms, essentially by factoring them out; if the terms are given by a fraction, this usually means dividing top and bottom by the dominant term. The main dominance relations to remember are:

$$\lim_{n\longrightarrow\infty} \frac{(\ln n)^p}{n^q} = 0, \quad \lim_{n\longrightarrow\infty} \frac{n^p}{e^{qn}} = 0, \quad \lim_{n\longrightarrow\infty} \frac{e^{pn}}{(n!)^q} = 0, \quad \lim_{n\longrightarrow\infty} \frac{(n!)}{n^n} = 0$$

for any $p, q > 0$. However, one has to be careful with the dominant terms if there are minus signs between them. For example, while the dominant term of $a_n = \sqrt{9^n+2^n} - 3^n$ may appear to be $3^n = \sqrt{9^n}$, in fact

$$a_n = \left(\sqrt{9^n+2^n} - 3^n\right) \cdot \frac{\sqrt{9^n+2^n} + 3^n}{\sqrt{9^n+2^n} + 3^n} = \frac{2^n}{\sqrt{9^n+2^n} + 3^n} = \left(\frac{2}{3}\right)^n \cdot \frac{1}{\sqrt{1 + (2/9)^n} + 1};$$

so the dominant term is $(2/3)^n$ (times 1/2, which does not effect anything). So the sequence a_n converges to 0, while the series $\sum_{n=1}^{\infty} a_n$ converges to something positive by the *Limit Comparison Test* applied with $b_n = (2/3)^n$.

If the terms of a sequence or series naturally split as a sum of two terms, one of which gives rise to a *convergent* sequence or series, respectively, then you can drop the convergent term in determining whether the entire sequence or series converges. For example, the sequence $a_n = (1 + (-1)^n)/n$ converges if and only if the sequence $b_n = (-1)^n/n$ does, because the sequence $c_n = 1/n$ converges (to 0, which does not matter in this case); so the sequence a_n does converge (to 0). Similarly, the series $\sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n}$ converges if and only if the series $\sum_{n=1}^{\infty} \frac{1}{n}$ does because the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the *Alternating Series Test*. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, neither does the series $\sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n}$. However, be careful *not* to split off a *divergent* sequence or series. For example,

$$\lim_{n\longrightarrow\infty} \left(\sqrt{9^n+2^n} - 3^n\right) \neq \lim_{n\longrightarrow\infty} \sqrt{9^n+2^n} - \lim_{n\longrightarrow\infty} 3^n;$$

$$\sum_{n=1}^{\infty} \left(\sqrt{9^n+2^n} - 3^n\right) \neq \sum_{n=1}^{\infty} \sqrt{9^n+2^n} - \sum_{n=1}^{\infty} 3^n;$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \neq \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}$$

because neither of the two limits on the right-hand side on the first line exists and neither of the four sums on the right-hand side of the second and third lines exists.

Convergence/divergence of sequences. A sequence is simply an infinite string of numbers described in some way, typically by an explicit formulas, such as $a_n = (-1)^n n^4/(3n^4+1)$, or by a recursive formula, such as $a_{n+1} = \sqrt{6+a_n}$, with some initial condition(s), such as $a_1 = \sqrt{6}$. While *sequence* is a longer word than *series*, determining whether a sequence converges or diverges is easier.

- If a sequence is given by an explicit formula, it is usually possible to determine whether it converges through a quick inspection. Begin by splitting it into parts if possible (often not; be careful) and determining the dominant term; see above. For example,

  $$a_n = (-1)^n \frac{n^4}{3n^4+1} = (-1)^n \frac{1}{3 + 1/n^4};$$

  so the dominant term here is $(-1)^n$. If plugging in $n = \infty$ makes sense then, you are done: the sequence converges; for example, it makes to plug in $n = \infty$ into $1/(3+1/n^4)$, but not into

$n^4/(3n^4+1)$ or $(-1)^n/(3+1/n^4)$, because ∞/∞ and $(-1)^\infty$ do not make sense. Typically, a sequence would not converge due to either an oscillatory behavior, which may be exhibited by a factor of $(-1)^n$ or $\sin(n)$, or because it (or part of it) approaches ∞, as $n/(\ln n)$ does. However, the presence of an oscillatory factor does not insure divergence; for example, the sequence

$$(-1)^n \frac{n^3}{3n^4+1} = \frac{(-1)^n}{n} \cdot \frac{1}{3+1/n^4}$$

converges to 0 because the seemingly oscillatory factor in fact decays to 0. Occasionally (if terms like 2^n, $n!$, or n^n are present), something like a *Ratio Test for Sequences* may be useful:

- if $\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = L$ and $|L| < 1$, then the sequence $\{a_n\}$ converges to 0;
- if $\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|}$ and $|L| > 1$ or $|a_{n+1}|/|a_n| \longrightarrow \infty$, then the sequence $\{a_n\}$ diverges.

This test however is not stated in the book. The first case follows from the *Ratio Test* for Series and the *most important divergence test* for series. However, in practice, the implication goes from the *Ratio Test for Sequences* to the *Ratio Test for Series.*

- If a sequence is given by a recursive formula, begin by writing the first few terms to get an idea whether sequence converges or diverges. If it appears to converge, the *Monotonic Sequence Theorem* may be useful to justify this (so you may need to use induction to show that either the sequence is bounded above by something and increasing or bounded below and decreasing). If it appears to diverge, this is likely due to some oscillatory behavior or because of going off to infinity; you'll need to justify that this pattern continues as n increases.
- The *Squeeze Theorem for Sequences* may be useful in some cases, but is generally avoidable. In some cases, it may be possible to replace n by x and compute the limit as $x \longrightarrow \infty$; this may allow using l'Hospital's rule (if the required conditions are satisfied), but usually this will not be the fastest approach.

Convergence/divergence of series. A series is the sum of terms in a sequence, with the latter typically given by an explicit formula when series are encountered. While *series* is a shorter word than *sequence*, determining whether a series converges is much harder and the concept of a series itself is significantly more abstract. First, a series $\sum_{n=1}^{\infty} a_n$ converges if and only if the *sequence of partial sums* $\{s_n\}$ defined by

$$s_n = a_1 + a_2 + \ldots + a_n$$

does; if this happens, the infinite sum of the a_n's is defined to be the limit of the s_n's. What this means is that you keep on adding more and more terms a_n to the sum and see if the resulting sums approach anything. However, in practice, it is almost never possible to find an explicit formula for s_n. Second, there are 7 divergence/convergence tests for series, most with several assumptions that you have to remember to check. After trying to split off a convergent part of a series (e.g. $\sum 1/n^2$ from $\sum(1/n^2 + (\sin n)/n^3)$) and determining the dominant term, you might want to try doing the following to determine if the series converges:

(0) if the sequence $\{a_n\}$ does not converge to 0, the series $\sum a_n$ diverges. For example, the series

$$\sum_{n=1}^{\infty}(-1)^n, \quad \sum_{n=1}^{\infty}\frac{n}{2n+1}, \quad \sum_{n=1}^{\infty}\cos(1/n), \quad \sum_{n=1}^{\infty}\sin(n),$$

all diverge. Note that even if $\lim\limits_{n\longrightarrow 0} a_n = 0$, the series $\sum a_n$ may still diverge; this is the reason you need the other half-dozen convergence/divergence tests.

(1) if the series is a geometric series $\sum cr^n$ or p-series $\sum 1/n^p$, you should know immediately if it converges or diverges (but do not confuse these with other similarly looking series; these two types of series are very restrictive, but also very important);

(2) if the series involves n in the exponent, e.g. 5^n (but not just n^5), n^n, $n!$, or more generally products with the number of factors increasing with n, try the general *Ratio Test*.

(3) if the series has **positive terms only**, determine its leading term, such as some power of n, and apply the *Limit Comparison Test* with that power of n. Remember that $\sin(1/n)$ and $\tan(1/n)$ look like $1/n$ as $n \longrightarrow \infty$, since

$$\lim_{n\longrightarrow\infty}\frac{\tan(1/n)}{1/n} = \lim_{n\longrightarrow\infty}\frac{\sin(1/n)}{1/n}\cdot\lim_{n\longrightarrow\infty}\cos(1/n) = \lim_{x\longrightarrow 0}\frac{\sin(x)}{x}\cdot 1 = 1.$$

So by the *Limit Comparison Test* with $b_n = 1/n^p$, the series

$$\sum_{n=1}^{\infty}\sin^p(1/n), \qquad \sum_{n=1}^{\infty}\tan^p(1/n)$$

converge if and only if $p > 1$. However, $\sin(n)$ and $\tan(n)$ do not look like n as $n \longrightarrow \infty$. If the *Limit Comparison Test* is not suitable, try to find a way to use the *Comparison Test*; so you'll still need to guess b_n, but now the second sequence needs to satisfy different requirements (but still 3 of them). For example, the *Limit Comparison Testt* with $b_n = 1/n^2$ cannot be used for the series $\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2}$ because

$$\lim_{n\longrightarrow\infty}\frac{a_n}{b_n} = \lim_{n\longrightarrow\infty}\frac{|\sin n|/n^2}{1/n^2} = \lim_{n\longrightarrow\infty}|\sin n|$$

does not exists. However, we can use the *Comparison Test* with $b_n = 1/n^2$, because

$$0 \le a_n = \frac{|\sin n|}{n} \le b_n = \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges; this implies that so does the "smaller" series $\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2}$. This argument cannot be used to directly conclude that the series $\sum_{n=1}^{\infty}\frac{|\sin n|}{n}$ diverges[3], because the divergence of the series $\sum_{n=1}^{\infty}\frac{1}{n}$ does not imply that the "smaller" series $\sum_{n=1}^{\infty}\frac{|\sin n|}{n}$ also diverges.

[3]this series does indeed diverge because $|\sin x| + |\sin(x+1)| \ge 1/2$ for all x

Finally, you can try to see if the *Integral Test* is applicable. For this, the function f obtained from the terms of the series by replacing n by x must make sense for all $x \geq 1$ (or at least for $x \geq N$ for some N); for example, $x!$ does not make sense. You also have to check that the function f obtained in this way is positive, continuous, and decreasing for $x \geq 1$ (or at least for $x \geq N$ for some N). For example, while the function $f(x) = |\sin x|/x$ makes sense for $x \geq 1$ and is continuous, it is not decreasing (and or even positive); so the fact that the integral $\int \frac{|\sin x|}{x} dx$ diverges does not say anything directly about the infinite series. The most important use of the *Integral Test* has been to obtain the *p-Series Test*; it has also been used in the present of $\ln n$. The *Integral Test* can be used to show that all of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^p}, \quad \sum_{n=1}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)^p}, \quad \ldots, \quad \sum_{n=1}^{\infty} \sin^p(1/n), \quad \sum_{n=1}^{\infty} \tan^p(1/n)$$

converge if and only if $p > 1$. Except for the last 2 series, the relevant integral can actually be computed fairly easy. In the case of the last 2 series, the integral is much harder to compute, but it can be shown to be finite if and only if $p > 1$, which suffices; however, it is simpler to apply the *Limit Comparison Test* to the last 2 series with $b_n = 1/n^p$.

(4) if the series has **terms of different signs**, first try to see if the series meets all three requirements of the *Alternating Series Test*; satisfaction of the alternating sign requirement is likely to be indicated by the presence of a factor of $(-1)^n$, but even so do not forget to check the other two conditions (and make sure to state them). If the *Alternating Series Test* does not apply, try the *Absolute Convergence Test*; this may allow you to apply one of the tests suitable only for series with non-negative terms. The *Alternating Series Test* is applicable to the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, but not to the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$; the *Absolute Convergence Test* is applicable to the second series, but not to the first. Both tests are applicable to the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, but neither to the series $\sum_{n=1}^{\infty} \frac{\sin n}{n}$. The only conclusion you can ever draw from either of these tests is that the series converges; if you want to show that a series with terms of different signs diverges, you need to find some other reason.

(5) in rare cases, it is possible to determine whether a series converges or diverges by **computing the corresponding sequence of partial sums**. This can be done when the series has the form

$$\sum_{n=1}^{\infty} (b_n - b_{n+m})$$

for some sequence $\{b_n\}$. If $n \geq m$, the n-th partial sum is then

$$\begin{aligned} s_n = s_1 + s_2 + \ldots + s_n &= (b_1 - b_{1+m}) + (b_2 - b_{2+m}) + \ldots + (b_n - b_{n+m}) \\ &= \sum_{k=1}^{k=m} b_k - \sum_{k=n+1}^{k=n+m} b_k, \end{aligned} \tag{22}$$

since the second term in the k-th pair cancels with the first term in the $(k+m)$-th, provided $k \leq n-m$; this leaves the first terms in the first m pairs and the second terms in the last m pairs.

As $n \longrightarrow \infty$, the first sum on the second line in (22) does not change; so the sequence $\{s_n\}$ (and thus the series $\sum_{n=1}^{\infty}\big(b_n - b_{n+m}\big)$) converges if and only if the sequence $\sum_{k=n+1}^{k=n+m} b_k$ does. This happens if the sequence $\{b_n\}$ converges, but may happen even if $\{b_n\}$ diverges. For example, all of the series

$$\sum_{n=1}^{\infty}\big(\sin(1/n) - \sin(1/(n+1))\big), \quad \sum_{n=1}^{\infty}\big(\cos(1/n) - \cos(1/(n+2))\big), \quad \sum_{n=1}^{\infty}(-1)^n\big(\ln(n) - \ln(n+2)\big)$$

converge, while the series

$$\sum_{n=1}^{\infty}\big(\cos(n) - \cos(n+1)\big), \quad \sum_{n=1}^{\infty}\big(\ln(n) - \ln(n+1)\big), \quad \sum_{n=1}^{\infty}\big(\mathrm{e}^n - \mathrm{e}^{n+1}\big)$$

diverge. This kind of cancellation is also useful for computing sums of series like $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ via partial fractions and partial sums. However, for showing that this series converges, it is much simpler to use the *Limit Comparison Test*.

Most importantly, try to see what a given series looks like, in terms of the leading terms and oscillatory behavior if any; in most cases, you may be able to guess whether it converges or diverges rather quickly based on these. If you are asked to justify your answer, make sure you check that all of the conditions of the test you want to use hold; often this will mean stating the required properties, but sometimes additional justification may be required. For example, it is sufficient to state that $1/n \geq 0$, but some explanation is required to justify that $1/(n^2 - n + 1) \geq 0$.

Remark: The *Integral Test* for series is a consequence of the definition of integral. The *Alternating Series Test* is a consequence of the *Monotonic Sequence Theorem*, which in turn is a fundamental statement about completeness of *real* (but not rational) numbers ("no holes" in the real numbers). The *Comparison Test for Series* is a consequence of the *Squeeze Theorem for Sequences*. The *Limit Comparison Test*, *Ratio Test* for positive sequences, and *Absolute Convergence Test* are consequences of the *Comparison Test*. The convergence statement the general *Ratio Test* is a consequence of the *Ratio Test* for positive sequences and the *Absolute Convergence Test*; its divergence statement follows from the *most important divergence test* for series. So, in principle, whenever *Limit Comparison Test*, *Ratio Test*, or *Absolute Convergence Test* is usable, so is the *Comparison Test* (the *Integral Test* and the *Alternating Series Test* are fundamentally different). However, in practice, whenever either *Limit Comparison Test*, *Ratio Test*, or *Absolute Convergence Test* is usable, it might be much easier to use them than the *Comparison Test*; for example, while it might be easy to guess a limit-compare-to sequence $\{b_n\}$, it may be harder to determine a suitable compare-to sequence $\{b_n\}$.

Good luck on the final exam