MAT 127: Calculus C, Fall 2009

Solutions to Mini-Quiz 1:

qualitative properties of 1st-order differential equations

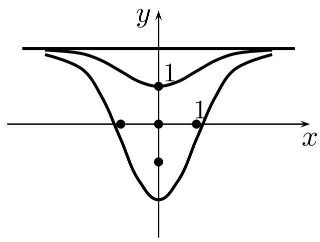
Problem A

Consider the four differential equations for $y = y(x)$:

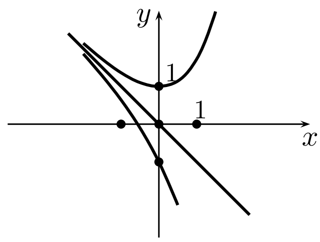
$$\text{(a) } y' = 2 - y, \qquad \text{(b) } y' = x(2 - y), \qquad \text{(c) } y' = x + y - 1 \qquad \text{(d) } y' = \sin x \sin y.$$

Each of the four diagrams below shows three solution curves for one of these equations:

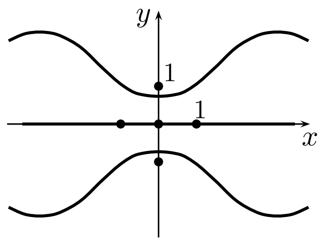
I



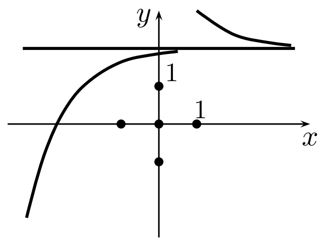
II



III



IV



Match each of the diagrams to the corresponding differential equation (the match is one-to-one) and explain your reasoning:

diagram	I	II	III	IV
equation	b	c	d	a

- I and IV are not (c) or (d) because the constant function $y = 2$ is not a solution of (c) or (d)
- I is not (a) because some solution curves in I descend when $y < 2$ (and $x < 0$); IV is not (b) because one of the solution curves in IV ascends when $y < 2$ and $x < 0$
- III is not (a), (b), or (c) because the constant function $y = 0$ is not a solution of (a), (b), or (c)
- II is not (a), (b), or (d) because the function $y = -x$ is not a solution of (a), (b), or (d); also the curves in II do not have zero slope for $y = 2$ or $y = 0$

Problem B: 7.2 3-6, p511

diagram	I	II	III	IV
equation	4	6	3	5

- I is not 3 because the slopes in I depend on x (change under horizontal shifts); I is not 5 or 6 because the slopes in I are horizontal for $y = 2$
- II is not 3, 4, or 5 because the slopes in II are horizontal for $y = 0$; II is not 3 also because the slopes in II depend on x
- III is not 4, 5, or 6 because the slopes in III do not depend on x (do not change under horizontal shifts)
- IV is not 3, 4, or 6 because the slopes are not 0 for either $y = 2$ or $y = 0$ (at least on the y-axis, where $x = 0$)

Remark: The above justifications, for Problems A and B, contain 12 elimination statements: each of the 4 diagrams is shown to be incompatible with 3 of the equations. Since you know that the match is one-to-one, it is possible to fully justify the answer with just 6 elimination statements, provided they are chosen properly. For example, after you match diagram I with equation 4 in Problem B, you can forget about equation 4 when considering the remaining 3 diagrams.

Solutions to Mini-Quiz 2:

convergence/divergence of sequences/series

Determine whether each of the following sequences or series converges or not. In each case, clearly circle either **YES** *or* **NO**, *but not both.*

(a) the sequence $a_n = \dfrac{(-1)^{n-1}n}{n^2+1}$ **YES** **NO**

Since $a_n = \dfrac{(-1)^n}{n} \cdot \dfrac{1}{1+1/n^2}$, this sequence converges to 0.

(b) the sequence $a_n = 1 + \cos(2/n)$ **YES** **NO**

Since $\cos(2/n) \longrightarrow \cos 0 = 1$, $a_n \longrightarrow 2$

(c) the sequence $a_n = n \cos n$ **YES** **NO**

Since $\cos n$ does not approach 0 as $n \longrightarrow 0$, $|a_n|$ takes arbitrarily large values

(d) the sequence $a_n = (-1)^n \dfrac{n}{n+1}$ **YES** **NO**

Since $a_n = (-1)^n \frac{1}{1+1/n}$, the odd terms approach -1, while the even terms approach 1.

(e) the sequence $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$ **YES** **NO**

Since $|\sin 2n| \leq 1$, while $\sqrt{n} \longrightarrow \infty$, $a_n \longrightarrow 0$

(f) the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$ **YES** **NO**

$\dfrac{n}{\sqrt{n^2+1}}$ looks like $\dfrac{n}{\sqrt{n^2}} = 1$: $\dfrac{n}{\sqrt{n^2+1}} = \dfrac{1}{\sqrt{n^2+1}/\sqrt{n^2}} = \dfrac{1}{\sqrt{1+1/n^2}} \longrightarrow 1 \neq 0$; so the series diverges by the most important divergence test for series

(g) the series $\displaystyle\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$ **YES** **NO**

$\dfrac{4+3^n}{2^n}$ looks like $\dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ and the geometric series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ diverges:

$$\frac{(4+3^n)/2^n}{3^n/2^n} = \frac{4+3^n}{3^n} = 4/3^n + 1 \longrightarrow 1.$$

Alternatively, $\displaystyle\sum_{n=1}^{\infty} \frac{4+3^n}{2^n} = \sum_{n=1}^{\infty} \frac{4}{2^n} + \sum_{n=1}^{\infty} \frac{3^n}{2^n}$ and the 1st series on RHS converges, while the second diverges, and so the series on LHS also diverges (but diverges+diverges would imply nothing!).

Also, $0 \le \dfrac{3^n}{2^n} \le \dfrac{4+3^n}{2^n}$ and the "smaller" series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ diverges

(h) the series $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{n!}$ **YES** **NO**

The power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges to e^x for all x.

You can also use the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{7^{n+1}/(n+1)}{7^n/n!} = \frac{7^{n+1}}{7^n} \cdot \frac{n!}{(n+1)!} = 7 \cdot \frac{1}{n+1} \longrightarrow 0 < 1.$$

(i) the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ **YES** **NO**

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges; since $0 \le |\sin n|/2^n \le 1/2^n$, so does the series $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{2^n}$ by Comparison Test, and thus so does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ by the Absolute Convergence Test.

(j) the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ **YES** **NO**

$\dfrac{n}{n^2+1} = \dfrac{1}{n+1/n} \longrightarrow 0$, $\dfrac{1}{n+1/n} > \dfrac{1}{n+1+1/(n+1)}$, because

$$n+1+1/(n+1) - \big(n+1/n\big) > 1 - 1/n \ge 0\,,$$

and the series is alternating.

Solutions to Mini-Quiz 3:

approximating sums of infinite series

Explain why each of the following series converges. Then estimate its sum to within 2^{-10} *using the minimal possible number of terms, justifying your estimate; leave your answer as a simple fraction* p/q *for some integers* p *and* q *with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why.*

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges by the *p-Series Test*, since here $p=5>1$.

All terms in this series are positive, and the *Remainder Estimate for the Integral Test* Theorem, on p583, should be used to estimate its sum with the required precision after all assumptions are checked. Since $f(x)=1/x^5>0$ is continuous and decreasing on $[1,\infty)$,

$$\int_{m+1}^{\infty}\frac{1}{x^5}\mathrm{d}x<\sum_{n=1}^{\infty}\frac{1}{n^5}-\sum_{n=1}^{n=m}\frac{1}{n^5}=\sum_{n=m+1}^{\infty}\frac{1}{n^5}<\int_{m}^{\infty}\frac{1}{x^5}\mathrm{d}x$$

Since

$$\int_{m}^{\infty}\frac{1}{x^5}\mathrm{d}x=\int_{m}^{\infty}x^{-5}\mathrm{d}x=\frac{1}{-4}x^{-4}\bigg|_{m}^{\infty}=\frac{1}{4}m^{-4}=\frac{1}{4m^4},$$

we find that

$$\frac{1}{4(m+1)^4}<\sum_{n=m+1}^{\infty}\frac{1}{n^5}<\frac{1}{4m^4}.$$

Since we need the middle term to be at most $1/2^{10}=1/4^5$, by the second inequality $m=4$ works; by the first inequality $m=3$ would not work (with $m=3$, the middle term above is strictly greater than $1/4^5=2^{-10}$). So the required estimate is

$$\sum_{n=1}^{n=m}\frac{1}{n^5}=\sum_{n=1}^{n=4}\frac{1}{n^5}=\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}=\frac{3^5\cdot4^5+3^5\cdot2^5+4^5+3^5}{3^5\cdot4^5}$$

$$=\frac{243\cdot1024+243\cdot32+1024+243}{243\cdot1024}=\boxed{\frac{257,875}{248,832}}$$

This is an $\boxed{\text{under-estimate}}$ for the infinite sum, because the finite-sum estimate is obtained by dropping only positive terms from the infinite sum.

Remark 1: You can justify convergence using the *Integral Test*, after checking the required assumptions: since $f(x)=1/x^5>0$ is continuous and decreasing on $[1,\infty)$, the infinite series converges because

$$\int_{1}^{\infty}\frac{1}{x^5}\mathrm{d}x=\int_{1}^{\infty}x^{-5}\mathrm{d}x=\frac{1}{-4}x^{-4}\bigg|_{1}^{\infty}=\frac{1}{4}$$

converges.

With this approach, you do not need to re-check the three assumptions before applying the *Remainder Estimate for the Integral Test* Theorem.

Remark 2: If you were asked to estimate the sum within $1/1000$, according to the book's recipe you would still need to take $m=4$. This is the *smallest* value of m for which the book's upper-bound on the remainder of the infinite series is no greater than the required precision (with $m=3$, the upper-bound is $1/(4\cdot3^4)=1/324>1/1000$). In principle, the remainder is smaller than the upper bound, so that a smaller m could still work. But if the estimate is to be within $1/1000$, $m=3$ still cannot work because the lower bound for the estimate can be improved to

$$a_{m+1}+\int_{m+2}^{\infty}\frac{1}{x^5}\mathrm{d}x=\frac{1}{(m+1)^5}+\frac{1}{4(m+2)^4}.$$

For $m=3$, this is $881/640,000>1/1000$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ converges because this is an alternating series (odd terms are negative, even terms are positive), $1/n^5 \longrightarrow 0$ as $n \longrightarrow 0$, and $1/n^5 > 1/(n+1)^5$ for all n.

In this case, the *Alternating Series Estimation* Theorem, on p588, should be used to estimate the sum of the series; the three required assumptions have already been checked above. So,

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5} - \sum_{n=1}^{n=m} \frac{(-1)^n}{n^5}\right| = \left|\sum_{n=m+1}^{\infty} \frac{(-1)^n}{n^5}\right| < \left|a_{m+1}\right| = \frac{1}{(m+1)^5}.$$

Since we need the left term to be at most $1/2^{10} = 1/4^5$, by the inequality $m=3$ works. So the required estimate is

$$\sum_{n=1}^{n=m} \frac{(-1)^n}{n^5} = \sum_{n=1}^{n=3} \frac{(-1)^n}{n^5} = \frac{(-1)^1}{1^5} + \frac{(-1)^2}{2^5} + \frac{(-1)^3}{3^5} = \frac{-2^5 \cdot 3^5 + 3^5 - 2^5}{2^5 \cdot 3^5}$$

$$= \frac{-32 \cdot 243 + 243 - 32}{32 \cdot 243} = \boxed{-\frac{7565}{7776}}$$

This is an $\boxed{\text{under-estimate}}$ for the infinite sum, because the last term used is negative.

Remark 1: You can justify convergence using the *Absolute Convergence Test*, since the series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^5}\right| = \sum_{n=1}^{\infty} \frac{1}{n^5}$$

converges by the *p-Series Test*. With this approach, you need to check the three required assumptions before applying the *Alternating Series Estimation* Theorem though.

Remark 2: According to the book's recipe, you need to take $m=3$ as done above because this is the *smallest* value of m for which the book's upper-bound on the remainder of the infinite series is no greater than the required precision (with $m=2$, the upper-bound is $1/(2+1)^5 = 1/243 > 2^{-10}$). In principle, the remainder is smaller than the upper bound, so that a smaller m could still work. However, the remainder can be bounded below by the absolute value of the sum of the first two dropped terms (one of which is positive and one is negative); in this case, this bound is

$$\left|a_{m+1} + a_{m+2}\right| = \left|\frac{(-1)^{m+1}}{(m+1)^5} + \frac{(-1)^{m+2}}{(m+2)^5}\right| = \left|\frac{1}{(m+1)^5} - \frac{1}{(m+2)^5}\right| = \frac{(m+2)^5 - (m+1)^5}{(m+1)^5(m+2)^5}.$$

If $m=2$, this is $781/(243 \cdot 1024) > 2^{-10}$, so $m=2$ would not work.

General Remark: In both cases, the infinite series is estimated by the sum of the first m terms,

$$\sum_{n=1}^{n=m} a_n = a_1 + a_2 + \ldots + a_m.$$

The hard part is to choose m so that the absolute value of the remainder of the series

$$\sum_{n=m+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{n=m} a_n$$

does not exceed the specified precision. For this, the m-th error is bounded above using either *Remainder Estimate for the Integral Test* Theorem, on p583, or *Alternating Series Estimation* Theorem, on p588, depending on the type of series to be estimated (most series are of neither type and cannot be estimated using these two theorems). The number m is then chosen so that the upper-bound on the m-th remainder term is no greater than the required precision.

Solutions to Mini-Quiz 4:

Taylor series and interval of convergence

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence.

(a) $f(x) = \dfrac{1}{1+9x^2}$ around $x = 0$

Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and this power series converges if $|x| < 1$,

$$\frac{1}{1+9x^2} = \frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-9)^n(x^2)^n = \boxed{\sum_{n=0}^{\infty}(-9)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n 9^n x^{2n}}$$

and this series converges whenever

$$|-9x^2| < 1 \iff x^2 < 1/9 \iff -1/3 < x < 1/3;$$

so the interval of convergence is $\boxed{(-1/3, 1/3)}$ and the radius is $\boxed{1/3}$

Remark: make sure to distribute the n in $(-9x^2)^n$, so that you end up with a power series in x, not a power series in $-9x^2$ or x^2; either of the two power series in the box is fine for the final answer though.

(b) $f(x) = \dfrac{x^2}{(1-2x)^2}$ around $x = 0$

Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and the interval of convergence of this power series is $(-1, 1)$,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \sum_{n=1}^{\infty} n x^{n-1}$$

and this power series still converges if and only if $|x| < 1$, since differentiation may only remove end-points from the interval of convergence. Thus,

$$\frac{x^2}{(1-2x)^2} = x^2 \cdot \frac{1}{(1-(2x))^2} = x^2 \sum_{n=1}^{\infty} n(2x)^{n-1} = x^2 \sum_{n=1}^{\infty} n 2^{n-1} x^{n-1} = \boxed{\sum_{n=1}^{\infty} n 2^{n-1} x^{n+1}}$$

and this series converges whenever

$$|2x| < 1 \iff -1/2 < x < 1/2;$$

so the interval of convergence is $\boxed{(-1/2, 1/2)}$ and the radius is $\boxed{1/2}$

Remark: make sure to distribute the $n-1$ in $(2x)^{n-1}$, so that you end up with a power series in x, not a power series in $2x$, and to multiple x^2 with x^{n-1}, so that you end up with a power series in x, not a product of such a power series with x^2.

(c) $f(x) = x^3$ around $x = -1$

In this case, we can compute all derivatives:

$$f^{\langle 0\rangle}(-1) = f(-1) = x^3|_{x=-1} = -1, \quad f^{\langle 1\rangle}(-1) = f'(-1) = 3x^2|_{x=-1} = 3,$$
$$f^{\langle 2\rangle}(-1) = f''(-1) = 6x|_{x=-1} = -6, \quad f^{\langle 3\rangle}(-1) = f'''(-1) = 6|_{x=-1} = 6,$$
$$f^{\langle n\rangle}(-1) = 0 \ \text{ if } n \geq 4.$$

Thus, by the main Taylor series formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{\langle n\rangle}(-1)}{n!}(x-(-1))^n = \frac{-1}{0!}(x+1)^0 + \frac{3}{1!}(x+1)^1 + \frac{-6}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$
$$= \boxed{-1 + 3(x+1) - 3(x+1)^2 + (x+1)^3}$$

Since this power series is a finite sum, it converges for all x (finitely many numbers can always be added together). Thus, the interval of convergence is $\boxed{(-\infty,\infty)}$ and the radius of convergence is $\boxed{\infty}$

Remark: in this case you can check your answer; the expression in the largest box equals

$$-1 + (3x+3) - (3x^2+6x+3) + (x^3+3x^3+3x+1) = x^3 \ \checkmark$$

However, make sure the answer you *give* is a polynomial (power series with finite number of terms) in $x+1$, not x.

(d) $f(x) = \mathrm{e}^{-x/2}$ around $x = 0$

Since $\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x,

$$\mathrm{e}^{-x/2} = \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}}$$

for all x. So the interval of convergence is $\boxed{(-\infty,\infty)}$ and the radius of convergence is $\boxed{\infty}$

Remark: make sure to distribute the n in $(-x/2)^n$, so that you end up with a power series in x, not a power series in $-x/2$.

Solutions to Mini-Quiz 5:

Computing sums of infinite series

Show that the following series are convergent and find their sums.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}x^n \Big|_{x=\frac{1}{2}}$$

Now use standard power series to sum up the power series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x|<1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n}x^n + C = \int\left(\sum_{n=1}^{\infty} x^{n-1}\right)\mathrm{d}x = \int \frac{\mathrm{d}x}{1-x} = -\ln(1-x) + C' \quad \text{if } |x|<1$$

$$\Longrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n}x^n = -\ln(1-x) \quad \text{if } |x|<1$$

Since the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}x^n$ converges whenever $|x| < 1$ and its sum (in those cases) equals $-\ln(1-x)$, the evaluation of this power series at $x = 1/2$, i.e. the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$, also converges and equals

$$-\ln(1-1/2) = -\ln(1/2) = \boxed{\ln 2}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} nx^n \Big|_{x=\frac{1}{2}}.$$

Now use standard power series to sum up the power series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x|<1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\left(\sum_{n=0}^{\infty} x^n\right)' = x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x}{(1-x)^2} \quad \text{if } |x| < 1.$$

Since the power series $\displaystyle\sum_{n=1}^{\infty} nx^n$ converges whenever $|x| < 1$ and its sum (in those cases) equals $x/(1-x)^2$, the evaluation of this power series at $x = 1/2$, i.e. the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$, also converges and equals

$$\frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = \boxed{2}$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\Bigg|_{x=\pi/4}.$$

Since the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ converges for all x and its sum equals $\sin x$, the evaluation of this power series at $x = \pi/4$, i.e. the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$, also converges and equals

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (\ln 2)^n}{n!}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\ln 2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\Bigg|_{x=-\ln 2}.$$

Since the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x and its sum equals e^x, the evaluation of this power series at $x = -\ln 2$, i.e. the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!}$, also converges and equals

$$\mathrm{e}^{-\ln 2} = \mathrm{e}^{(-1)\ln 2} = \mathrm{e}^{\ln(2^{-1})} = 2^{-1} = \boxed{\frac{1}{2}}$$

Remark 1: The principle in all four cases is to write the infinite series as the evaluation of some power series in x at some value and then determine the function $f(x)$ to which the power series sums up. The latter is easy in (c) and (d), because the power series there are two of the standard ones; in (a) and (b) the power series are obtained from a standard power series by integration and differentiation, respectively. Once $f(x)$ is determined, simply plug in the correct value of x to get the sum of the infinite series. You can show that the infinite series itself converges by checking that "the correct value of x" lies in the interval of convergence of the power series.

Remark 2: In all of the above cases, you can show that the infinite series converges by using one of the convergence tests. Since exponents of n are involved (2^n, π^{2n}, $(\ln 2)^n$), the Ratio Test is the first test to try (and it works in all four cases). For the series in (c) and (d) you can also use the Alternating Series Test. If you check convergence of the infinite series in this way, you no longer need to worry about the interval of convergence of the power series.

Solutions to Mini-Quiz 6:

Taylor series, limits, and integration

(a) *Find the radius and interval of convergence of the power series*

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{1+\sqrt{n}}.$$

To find the radius of convergence, use the Ratio Test with $a_n = 2^n x^n/(1+\sqrt{n})$:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{2^{n+1}|x|^{n+1}/(1+\sqrt{n+1})}{2^n|x|^n/(1+\sqrt{n})} = 2|x|\cdot\frac{1+\sqrt{n}}{1+\sqrt{n+1}} = 2|x|\cdot\frac{1/\sqrt{n}+\sqrt{n}/\sqrt{n}}{1/\sqrt{n}+\sqrt{n+1}/\sqrt{n}}\\ &= 2|x|\cdot\frac{1/\sqrt{n}+1}{1/\sqrt{n}+\sqrt{(n+1)/n}} = 2|x|\cdot\frac{1/\sqrt{n}+1}{1/\sqrt{n}+\sqrt{1+1/n}} \longrightarrow 2|x|\cdot\frac{0+1}{0+\sqrt{1+0}} = 2|x|.\end{aligned}$$

So the series converges if $2|x|<1$ and diverges if $2|x|>1$. Thus, the radius of convergence is $\boxed{1/2}$ and it remains to check convergence for $x=\pm 1/2$. For $x=1/2$, we get the series

$$\sum_{n=0}^{\infty} \frac{2^n(1/2)^n}{1+\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}.$$

This series has only positive terms, which look like $1/\sqrt{n} = 1/n^{1/2}$:

$$\frac{a_n}{b_n} = \frac{1/(1+\sqrt{n})}{1/\sqrt{n}} = \frac{\sqrt{n}}{1+\sqrt{n}} = \frac{\sqrt{n}/\sqrt{n}}{1/\sqrt{n}+\sqrt{n}/\sqrt{n}} = \frac{1}{1/\sqrt{n}+1} \longrightarrow \frac{1}{0+1} = 1.$$

Since the series $\sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$ diverges by the p-Series Test (with $p=1/2 \le 1$), by the Limit Comparison Test so does the series $\sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}$. Thus, the end-point $x=1/2$ is not in the interval of convergence. For $x=-1/2$, we get the series

$$\sum_{n=0}^{\infty} \frac{2^n(-1/2)^n}{1+\sqrt{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}.$$

This series is alternating (odd terms are negative, even terms are positive), $1/(1+\sqrt{n}) \longrightarrow 0$, and $1/(1+\sqrt{n}) > 1/(1+\sqrt{n+1})$; so this series converges by the Alternating Series Test. Thus, the end-point $x=-1/2$ is in the interval of convergence. So, the interval of convergence is $\boxed{[-1/2, 1/2)}$

Remark 1: remember that the Ratio Test is always used to determine the *radius* of convergence, but can **never** be used to determine whether a power series converges at the end-points of the interval of convergence; you need to use some other convergence test.

Remark 2: you cannot use the Comparison Test with $b_n = 1/\sqrt{n}$ above, because the divergence of the series $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ does not imply the divergence of the "smaller" series $\sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}$ via the Comparison Test (but it does imply divergence via the Limit Comparison Test).

(b) *Find* $\displaystyle\lim_{x\longrightarrow 0}\frac{f(x)-1-x}{1-e^{x^2}}$

Since $\displaystyle e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$,

$$e^{x^2}=\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}\qquad\Longrightarrow\qquad 1-e^{x^2}=1-\left(1+\frac{x^2}{1!}+\frac{x^4}{2!}+\dots\right)=-\frac{x^2}{1!}-\frac{x^4}{2!}-\dots$$

Plugging the power series back into the fraction gives

$$\begin{aligned}\frac{f(x)-1-x}{1-e^{x^2}}&=\frac{\left(1+\frac{2x}{2}+\frac{4x^2}{1+\sqrt{2}}+\frac{8x^3}{1+\sqrt{3}}+\dots\right)-1-x}{-\frac{x^2}{1!}-\frac{x^4}{2!}-\dots}=-\frac{\frac{4x^2}{1+\sqrt{2}}+\frac{8x^3}{1+\sqrt{3}}+\dots}{x^2+\frac{x^4}{2}+\dots}\\&=-\frac{\left(\frac{4x^2}{1+\sqrt{2}}+\frac{8x^3}{1+\sqrt{3}}+\dots\right)/x^2}{\left(x^2+\frac{x^4}{2}+\dots\right)/x^2}\\&=-\frac{\frac{4}{1+\sqrt{2}}+\frac{8x}{1+\sqrt{3}}+\dots}{1+\frac{x^2}{2}+\dots}\underset{x\longrightarrow 0}{\longrightarrow}-\frac{\frac{4}{1+\sqrt{2}}+0+0+\dots}{1+0+0+\dots}=-\frac{4}{\sqrt{2}+1}=\boxed{-4(\sqrt{2}-1)}\end{aligned}$$

Remark: l'Hospital's rule can also be used here, but it has to be applied twice with the assumptions checked each time, as follows. Since

$$\lim_{x\longrightarrow 0}\big(f(x)-1-x\big)=f(0)-1=0\qquad\text{and}\qquad\lim_{x\longrightarrow 0}\big(1-e^{x^2}\big)=1-e^0=0,$$

by l'Hospital's rule

$$\lim_{x\longrightarrow 0}\frac{f(x)-1-x}{1-e^{x^2}}=\lim_{x\longrightarrow 0}\frac{\big(f(x)-1-x\big)'}{\big(1-e^{x^2}\big)'}=\lim_{x\longrightarrow 0}\frac{f'(x)-1}{-2xe^{x^2}}=-\frac{1}{2}\lim_{x\longrightarrow 0}\frac{f'(x)-1}{xe^{x^2}}.$$

Differentiating the power series for f, we get

$$f'(x)=\sum_{n=1}^{\infty}\frac{2^n n x^{n-1}}{1+\sqrt{n}}=\frac{2}{1+\sqrt{1}}+\frac{2\cdot 2^2 x}{1+\sqrt{2}}+\dots.$$

Since

$$\lim_{x\longrightarrow 0}\big(f'(x)-1\big)=f'(0)-1=0\qquad\text{and}\qquad\lim_{x\longrightarrow 0}\big(xe^{x^2}\big)=0e^0=0,$$

by l'Hospital's rule

$$\lim_{x\longrightarrow 0}\frac{f'(x)-1}{xe^{x^2}}=\lim_{x\longrightarrow 0}\frac{\big(f'(x)-1\big)'}{\big(xe^{x^2}\big)'}=\lim_{x\longrightarrow 0}\frac{f''(x)}{e^{x^2}+2x^2e^2}=\frac{f''(0)}{1+0e^0}.$$

Differentiating the power series for f', we get

$$f''(x)=\sum_{n=2}^{\infty}\frac{2^n n(n-1)x^{n-2}}{1+\sqrt{n}}=\frac{8}{1+\sqrt{2}}+\dots,$$

and so $f''(0)=8/(1+\sqrt{2})$. Putting this all together gives

$$\lim_{x\longrightarrow 0}\frac{f(x)-1-x}{1-e^{x^2}}=-\frac{1}{2}\lim_{x\longrightarrow 0}\frac{f'(x)-1}{xe^{x^2}}=-\frac{1}{2}f''(0)=-\frac{4}{1+\sqrt{2}}=\boxed{-4(\sqrt{2}-1)}$$

as in the first approach.

(c) *Find the Taylor series expansion for the function $g = g(x)$ given by*

$$g(x) = \int_0^x \frac{f(u) - 1}{u} \mathrm{d}u$$

around $x = 0$. What are the radius and interval of convergence of this power series?

Since

$$\frac{f(u) - 1}{u} = u^{-1} \sum_{n=1}^{\infty} \frac{2^n u^n}{1 + \sqrt{n}} = \sum_{n=1}^{\infty} \frac{2^n u^{n-1}}{1 + \sqrt{n}},$$

we get

$$g(x) = \int_0^x \frac{f(u) - 1}{u} \mathrm{d}u = \sum_{n=1}^{\infty} \frac{2^n u^n}{n(1 + \sqrt{n})} \Big|_{u=0}^{u=x} = \sum_{n=1}^{\infty} \frac{2^n x^n}{n(1 + \sqrt{n})}.$$

Since integration does not change the radius of convergence of a power series, the radius of convergence of the power series for g is still $\boxed{1/2}$ Since integration can only add end-points to the interval of convergence, the end-point $x = -1/2$ is still in the interval of convergence. So we only need to check whether the power series converges for $x = 1/2$. Replacing x with $1/2$, we get

$$\sum_{n=1}^{\infty} \frac{2^n (1/2)^n}{n(1 + \sqrt{n})} = \sum_{n=1}^{\infty} \frac{1}{n(1 + \sqrt{n})}$$

This series has only positive terms, which look like $1/n\sqrt{n} = 1/n^{3/2}$:

$$\frac{a_n}{b_n} = \frac{1/(n(1+\sqrt{n}))}{1/(n\sqrt{n})} = \frac{\sqrt{n}}{1 + \sqrt{n}} = \frac{\sqrt{n}/\sqrt{n}}{1/\sqrt{n} + \sqrt{n}/\sqrt{n}} = \frac{1}{1/\sqrt{n} + 1} \longrightarrow \frac{1}{0 + 1} = 1.$$

Since the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-Series Test (with $p = 3/2 > 1$), by the Limit Comparison Test so does the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n(1 + \sqrt{n})}$. Thus, the end-point $x = 1/2$ is in the interval of convergence. So, the interval of convergence is $\boxed{[-1/2, 1/2]}$

Remark 1: In this case, the Comparison Test with $b_n = 1/n^{3/2}$ can be used because the convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n\sqrt{n}}$ implies the convergence of the "smaller" series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n(1+\sqrt{n})}$ via the Comparison Test.

Remark 2: if you do not want to use the fact that integration of a power series can only add end-points to the interval of convergence (and differentiation can only remove them), then you can just proceed as in part (a). First, compute the radius of convergence using the Ratio Test, but now with $a_n = 2^n x^n/(n(1+\sqrt{n}))$; this change in a_n does not actually change the limit of $|a_{n+1}|/|a_n|$. Once you determine that the radius of convergence is $1/2$, check whether the powers series converges at each of the end-points, $x = 1/2$ and $x = -1/2$.