MAT 127: Calculus C, Fall 2009
Course Summary II

Extremely Important: sequences vs. series (do not mix them or their convergence/divergence tests up!!!); what it means for a sequence or series to converge or diverge; systems of 2 autonomous first-order differential equations and phase-plane portraits

Very Important: convergence/divergence tests for sequences and series; equilibrium/stationary points for systems of 2 autonomous first-order differential equations

Important: limit rules for sequences and series; computing limits of convergent sequences and sums of convergent series; sketching graphs of a solution to a system of 2 autonomous first-order differential equations as functions of time from phase trajectory and vice versa

Systems of 2 Autonomous First-order Differential Equations

(1) A system of 2 autonomous first-order differential equations is a system of the form

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \qquad (x,y) = \big(x(t), y(t)\big), \tag{1}$$

where f and g are some functions of x and y. For example,

$$\begin{cases} \frac{dx}{dt} = f(x,y) = \frac{1}{20}x - \frac{1}{500}xy \\ \frac{dy}{dt} = g(x,y) = -\frac{1}{10}y + \frac{1}{1000}xy \end{cases} \qquad (x,y) = \big(x(t), y(t)\big). \tag{2}$$

The systems (1) and (2) are called autonomous because they do not involve t explicitly. A solution of such a system is a pair of functions $(x,y) = (x(t), y(t))$ which satisfy both equations at the same time; neither $x = x(t)$ nor $y = y(t)$ separately is a solution (of the system). In order to check that a given pair of functions solves a system, simply compute LHS and RHS of the first equation for the two given functions and check that they are equal, and then compute LHS and RHS of the second equation for the two given functions and check that they are equal. This is usually not difficult; actually finding such pairs of functions is difficult.

(2) The first step in analyzing the system (1) is to find the constant solutions or the equilibrium points of (1). These are the points (x_i, y_i) in the xy-plane such that each constant function $(x(t), y(t)) = (x_i, y_i)$ is a solution of (1). The physical interpretation of this is that if the system starts at an equilibrium point, it stays there forever. In mathematical terms, this means that if the initial value $(x(0), y(0))$ of a solution $(x,y) = (x(t), y(t))$ to (1) is an equilibrium point, then $(x(t), y(t)) = (x_0, y_0)$ for all t. Since the derivative of a constant function is zero, the constant function $(x(t), y(t)) = (x_i, y_i)$ is a solution of (1) if and only if $f(x_i, y_i) = (0,0)$ and $g(x_i, y_i) = (0,0)$. Thus,

$$\boxed{(x_i, y_i) \text{ is equilibrium pt for } \begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \quad (x,y) = (x(t), y(t)) \quad \Longleftrightarrow \quad \begin{cases} f(x_i, y_i) = 0 \\ g(x_i, y_i) = 0 \end{cases}}$$

Thus, in order to find the equilibrium points for (1) or constant solutions of (1), we only need to solve the system

$$\begin{cases} f(x_i, y_i) = 0 \\ g(x_i, y_i) = 0 \end{cases}$$

This system does not involve any derivatives! For example, we find the equilibrium points for (2) by solving:

$$\begin{cases} f(x,y) = \frac{1}{20}x - \frac{1}{500}xy = 0 \\ g(x,y) = -\frac{1}{10}y + \frac{1}{1000}xy = 0 \end{cases} \iff \begin{cases} \frac{1}{20}x\big(1 - \frac{1}{25}y\big) = 0 \\ -\frac{1}{10}y\big(1 - \frac{1}{100}x\big) = 0 \end{cases} \iff \begin{cases} x=0 \text{ or } y=25 \\ y=0 \text{ or } x=100 \end{cases} \tag{3}$$

Thus, the equilibrium points of the system (2) are $(0,0)$ and $(100,25)$; they are indicated by the two large dots on the first sketch in Figure 1.

WARNING: While it is usually not hard to find the equilibrium points of (1), some care is often needed. For example, after the last step in (3), we need to determine all pairs (x,y) such that one of the two conditions on the top line is satisfied, so that $\frac{\mathrm{d}x}{\mathrm{d}t}=0$, **and** one of the two conditions on the bottom line is satisfied, so that $\frac{\mathrm{d}y}{\mathrm{d}t}=0$. This is different from finding (x,y) such that any two of the four conditions in (3) are satisfied; so $(x,y)=(0,25)$ is *not* an equilibrium point. Thus, it is essential to keep the conditions for $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ and the condition for $\frac{\mathrm{d}y}{\mathrm{d}t}=0$ separately, e.g. on separate lines.

(3) We are interested in knowing what happens with the point $(x(t),y(t))$, where $(x,y)=(x(t),y(t))$ is a solution of (1), as t increases. One special property of systems of autonomous equations is that if $(x,y)=(x(t),y(t))$ is a solution of such a system, e.g. of (1), then so is

$$(\tilde{x},\tilde{y}) = \big(\tilde{x}(t),\tilde{y}(t)\big) = \big(x(t-a),y(t-a)\big)$$

for any fixed constant a. As the time parameter t increases, the points $(x(t),y(t))$ and $(\tilde{x}(t),\tilde{y}(t))$ trace the same path in the xy-plane, but $(\tilde{x}(t),\tilde{y}(t))$ is delayed by time a. Thus, the behavior of a solution $(x(t),y(t))$ of (1) is well-represented by the directed curve in the xy-plane traced by $(x(t),y(t))$ as t increases. Such a curve is called is a phase trajectory for the system (1). It shows every point in the xy-plane as the path $(x(t),y(t))$ passes through as t increases, though it does not specify at what value of t the solution $(x,y)=(x(t),y(t))$ arrives at each given point (except possibly for $t=0$). Different phase trajectories for the same system generally do not intersect, but may converge at some point. It is generally much easier to find explicit xy-equations describing phase trajectories for a system of differential equation than actual solutions. These curves in the xy-plane (the trajectories, not solutions, which are functions, *not* curves) satisfy the differential equation obtained by dividing the second equation in (1) by the first and viewing y as a function of x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}, \qquad y = y(x).$$

Solutions to this equation in a specific case are analyzed in 7.6 using the direction field for the xy-differential equation. In the case of (2), we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{10}y + \frac{1}{1000}xy}{\frac{1}{20}x - \frac{1}{500}xy} = -\frac{y(100-x)}{2x(25-y)}, \qquad y = y(x).$$

This equation is separable and thus solvable. From this we find that the phase trajectories in the first quadrant of the xy-plane (not including the axes) are closed curves circling around the only equilibrium point in the first quadrant.

(4) Another way to represent a solution $(x,y)=(x(t),y(t))$ to the system (1) is by sketching the graphs of $x = x(t)$ and $y = y(t)$ as functions of time. It is most appropriate to do so with the

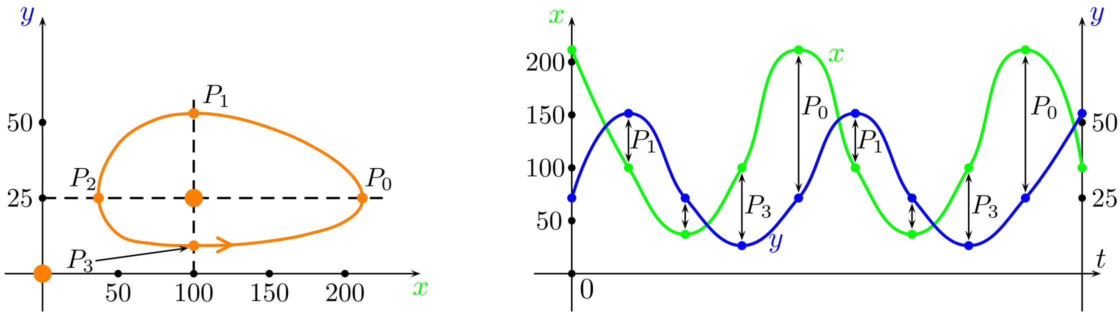

Figure 1: The left diagram shows a phase trajectory and the equilibrium points (the two large dots) for the system (2); a solution $(x(t),y(t))$ of (2) goes around this curve counter-clockwise as t increases. The right diagram shows the corresponding graphs of $x=x(t)$ and $y=y(t)$ as functions of time.

same horizontal t-axis, but different vertical x- and y-axes. The second diagram in Figure 1 shows rough graphs of the functions $x=x(t)$ and $y=y(t)$ for the solution $(x(t),y(t))$ of (2) whose phase trajectory is shown in the first diagram. The t-axis is shared by the two graphs, but the vertical axes are different. In particular, the x-axis has nothing to do with the y-graph; so $y(0)\approx 25$, not 75. Similarly, the y-axis has nothing to do with the x-graph; so at the last time point shown on the graph $x\approx 100$, and not 35. The intersection points of the two graphs are pretty much irrelevant as x and y may represent very different quantities, in addition to the x-axis and y-axes having different scales. What the sketch does tell us is the x-value(s) at the time(s) when y-value was something and vice versa, usually without specifying the corresponding value of time t. For example, the first time x was about 45 (the first x-min), y was about 25; because both graphs are periodic, $y\approx 25$ when x reaches about 45 for the second time. Both of these facts are indicated by the tiny vertical unlabeled line segments in the second diagram in Figure 1. This tell us that roughly $(45,25)$ should be a point on the corresponding phase trajectory (labeled P_2 on the first diagram in Figure 1). Furthermore, this is the left-most point on the trajectory (because this is the minimal value of x on the graph) and the trajectory passes through this point at least twice. Since a solution of (1) is determined by its value at $t=0$ (or any other fixed value of t), the latter implies that the corresponding trajectory keeps on going around a simple closed curve in the xy-plane.

(5) A central (and hard!) theme of 7.6 is to roughly sketch the graphs of $x=x(t)$ and $y=y(t)$ as functions of t from the phase trajectory traced by $(x,y)=(x(t),y(t))$ in the xy-plane and vice versa. In order to do so, first determine the extremal points of the phase trajectory or of the graphs (whichever you are given) and the limiting behavior if any. For example, the phase trajectory in the first diagram in Figure 1 has four extremal points that are traversed in the order $P_0,P_1,P_2,P_3,P_0,P_1,\ldots$; it has no limiting behavior (it keeps on circling around instead of approaching some point). If our trajectory instead spiraled down to the point (100,25), it would have had lots of extremal of points (one after each quarter-turn) and would have also approached (100,25) as $t\longrightarrow\infty$. The trajectory in the second sketch in Figure 2 limits to (1150,200) as $t\longrightarrow\infty$.

After determining the extremal points of a given phase trajectory, mark the x- and y-coordinates of each of them above the same t-point and do so in the order the extremal point are traverse. So if we start at P_0 in the first sketch in Figure 1, mark $x=210$ and $y=25$ above $t=0$ (however, remember that the x and y-scales may not be the same). After that, mark the x and y-coordinates of P_1, 100 and say 52, over some $t = t_1 > 0$. Then mark the x and y-coordinates of P_2, say 45 and 25, over some $t = t_2 > t_1$; continue on to P_3 and then $P_4 = P_0$. In order to indicate the periodic behavior of the trajectory, this should be done for at least one full period (so the x and y-coordinates of at least P_0 must be marked over two different t-values); it is preferable to continue for slightly longer. If the trajectory has a limiting point, the graphs should be done for long enough to indicate that they approach some asymptotes, such as in the first sketch in Figure 2. Make sure to distinguish between the x-points and the y-points (for example, use a pencil and a blue pen). Once you have marked the coordinates of the extremal points, connect the x-points by a smooth curve which is monotonic between any two of them and do the same for the y-points. To get a more precise sketch, you could also use coordinates of the non-extremal points on the phase trajectory, but using just the extremal points and the limiting behavior will suffice in most cases. Note that the t-axis should have no indication of scale; the only labels on it should be t on the very right and 0, provided the starting point of the trajectory is given. The scales of the x and y-axes and the labels on them in the "graphs sketch" should be analogous to the scales and the labels on these axes in the xy-plane.

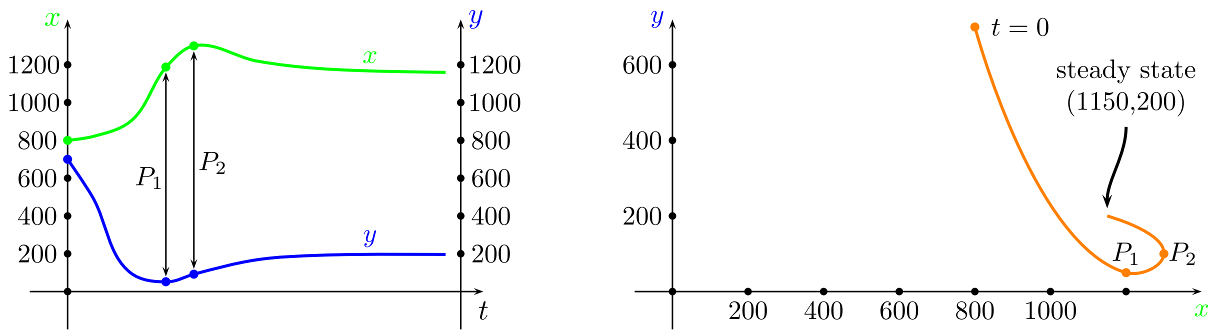

Figure 2: The left diagram shows graphs of some functions $x = x(t)$ and $y = y(t)$. The right diagram the phase trajectory trajectory traced by $(x, y) = (x(t), y(t))$ in the xy-plane.

If you start with graphs of $x = x(t)$ and $y = y(t)$, begin by marking the peaks and sags on each of the two graphs as well as the point on the other graph on the same vertical line as each of the peaks and sags. The x-values and y-values at these points give extremal points on the corresponding phase trajectory; you should also mark the starting point $(x_0, y_0) = (x(0), y(0))$. Once you have marked the extremal points in the xy-plane, connect them by a smooth curve in the order of increasing t so that the curve does not change its general direction (e.g. up and to the right) between any two of the points. If the graphs are periodic, the phase trajectory will be a closed curve. If the graphs have asymptotes, the phase trajectory will have a limiting point which is approached from the last extremal marked point (it is possible that there are infinitely many extremal points approaching the limiting point). For example, in the case of the first sketch in Figure 2, the starting values of x and y give the point $(800, 700)$ in the xy-plane. Then mark the sag on the y-curve along with the point on the x-curve directly above it and the peak on the x-curve along with the point on the y-curve directly below it. The x-value and y-value of the first pair give the point $P_1 \approx (1200, 50)$ in the xy-plane; the second pair gives the point $P_2 \approx (1300, 100)$ in the xy-plane. Connect the three points by a smooth curve in the xy-plane that does not change the general direction and after passing P_2 heads toward $(1150, 200)$; this is because the x-graph approaches $x = 1150$ and the y-graph approaches $y = 200$ as $t \longrightarrow \infty$. In the case of the first diagram in Figure 2, the x- and y-scales are the same, but usually this is not the case.

Note: The process of going between a phase trajectory and corresponding graphs, in either direction, does not require knowing the corresponding system of differential equations; for example, it was not specified on some of the homework exercises. Knowing the system may help check you have completed this process correctly. If $P_i = (x_i, y_i)$ is a horizontally extremal point of a phase trajectory for the system (1), then $f(x_i, y_i)=0$; if it is a vertically extremal point of a phase trajectory for the system (1), then $g(x_i, y_i)=0$. In the case of the system (2) and the first sketch in Figure 1, the former means that the y-coordinates of P_0 and P_2 are both 25; the latter means that the y-coordinates of P_1 and P_3 are both 100. Similarly, if the graph of $x=x(t)$ as a function of t has a peak or a sag at x_i and the y-value on the same vertical line is y_i, then $f(x_i, y_i)=0$; if the y-graph has a peak or a sag at y_i and the x-value on the same vertical line is x_i, then $g(x_i, y_i)=0$. In the case of the system (2) and the second sketch in Figure 1, the latter means that $y=25$ on a vertical line passing through an x-peak or x-sag; the former means that $x=100$ on a vertical line passing through an y-peak or y-sag.

(6) In 7.6, systems of 2 autonomous first-order differential equations are used to model interactions of two species. In such cases, $x(t)$ denotes the population of one of the species at time t, while $x(t)$ denotes the population of the other species. Such a system normally has an equilibrium point $(0,0)$ corresponding to no population of either species. With $y=0$, the $\frac{\mathrm{d}x}{\mathrm{d}t}$ equation in (1) describes the growth rate of the first species in the absence of the second; with $x = 0$, the $\frac{\mathrm{d}y}{\mathrm{d}t}$ equation in (1) describes the growth rate of the second species in the absence of the first. Each of these reduced equations is likely to be an exponential growth/decay equation or a logistic growth equation. In the exponential growth case, the population of the species increases exponentially in the absence of the other species; in the exponential decay case, the population decays out to 0 in the absence of the other species. In the logistic growth case, the population approaches the carrying capacity; this gives an equilibrium point $(K,0)$ or $(0,K)$, where K is the carrying capacity for the first population or the second population. There may well be other equilibrium points, with both populations nonzero; these correspond to the two populations precisely matched up to "support" each other (including possibly by one feeding on the other). The terms in the $\frac{\mathrm{d}x}{\mathrm{d}t}$ equation that involve y indicate whether the second species has a positive or negative effect on the first; the terms in the $\frac{\mathrm{d}y}{\mathrm{d}t}$ equation that involve x indicate whether the first species has a positive or negative effect on the second. From considering these terms, it should be possible to determine whether the system models a predator-prey relation $(+/-)$, that of cooperation for mutual benefit $(+/+)$, or of competition for common resources $(-/-)$.

In the case of (2),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{20}x \quad \text{if } y = 0, \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{10}y \quad \text{if } x = 0.$$

So, in the absence of the second species, the first obeys an exponential growth equation and thus increases exponentially with time; in the absence of the first species, the second obeys an exponential decay equation and thus eventually dies out. Since xy has a negative coefficient in the $\frac{\mathrm{d}x}{\mathrm{d}t}$ equation in (2) and positive in the $\frac{\mathrm{d}y}{\mathrm{d}t}$ equation, the presence of the second species has negative effect on the first and the presence of the first species has positive effect on the second. This suggests that the first species is prey and the second is predator.

Sequences

(1) A sequence is an infinite string of numbers. It can be specified in several ways:

- list the numbers; for example, $-1, 1/2, -1/6, 1/24, -1/120, \ldots$;
- give a formula for the n-th number in the sequence; for example $a_n = (-1)^n/n!$ for $n \geq 1$;
- through recursive definition; for example, $a_1 = -1$, $a_{n+1} = -a_n/(n+1)$ for $n \geq 1$, or $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. The first of these sequences is the same as the two sequences above; the second one is the famous Fibonacci sequence.

A sequence does not have to start with a_1; it could start with a_0 or with any other a_{n_0}, as long as a_n is specified for all $n \geq n_0$. Since it is just a string of numbers, the first number could be called a_1, or a_0, or a_{-10}; the second number in the sequence would then have to be called a_2, a_1, or a_{-9}, respectively.

(2) Given a sequence $a_1, a_2, \ldots$, we'd like to know whether it gets closer and closer to some number a or there is no such number. In the former case, the sequence is said to converge to a and this is written as $\lim_{n \longrightarrow \infty} a_n = \infty$; in the latter case, the sequence is said to diverge. In many cases, it is fairly straightforward to determine whether a sequence converges (and if so to what limit) or diverges. For example, if

$$a_n = (-1)^n \frac{\sqrt{n^4 + n^2}}{n^2 + \sqrt{n^2 + 1}},$$

simply divide top and bottom by n^2 (you have to divide by the same thing!):

$$\begin{aligned} a_n &= (-1)^n \frac{\sqrt{n^4 + n^2}/n^2}{n^2/n^2 + \sqrt{n^2 + 1}/n^2} = (-1)^n \frac{\sqrt{n^4/n^4 + n^2/n^4}}{1 + \sqrt{n^2/n^4 + 1/n^4}} \\ &= (-1)^n \frac{\sqrt{1 + 1/n^2}}{1 + \sqrt{1/n^2 + 1/n^4}}; \end{aligned} \tag{4}$$

so the fraction approaches $\sqrt{1}/(1 + \sqrt{0}) = 1$, but the sign alternates. So the terms a_n with n odd converge to -1, while the terms a_n with n even converge to 1; thus, the entire sequence *diverges* (there is no single number to which all of the terms approach). The above trick of dividing top and bottom of a fraction by a power of n is the **most common approach** to dealing with sequences given by fractions; in MAT 125, a similar trick involved dividing by a power of x. In order to reduce your chances of making minor computational errors, which may even alter the qualitative answer (and thus be heavily penalized), it is best not to skip steps in a computation like (4); in particular, be careful when taking a power of n under a square root.

(3) Some sequences are of the form $a_n = f(b_n)$ for some fairly simple function f and some fairly simple sequence b_n. For example, if $a_n = \mathrm{e}^{1/n}$, then the sequence $b_n = 1/n$ converges to 0 and since e^x is continuous at 0,

$$\lim_{n \longrightarrow \infty} \mathrm{e}^{1/n} = \mathrm{e}^{\lim_{n \longrightarrow \infty} 1/n} = \mathrm{e}^0 = 1.$$

On the other hand, the sequence $a_n = \cos(\pi n)$ is divergent, since it alternates between 1 and -1.

(4) If $a_n = f(n)$ for some function $f = f(x)$ defined on the positive real line, then

$$\lim_{n \longrightarrow \infty} a_n = \lim_{x \longrightarrow \infty} f(x)$$

if the second limit exists; the first limit may exist even if the second does not. This may allow using l'Hospital's rule for limits of functions. For example, if $a_n = (\ln n)/n$, then

$$\lim_{n \longrightarrow \infty} a_n = \lim_{x \longrightarrow \infty} \frac{\ln x}{x} = \lim_{x \longrightarrow \infty} \frac{(\ln x)'}{x'} = \lim_{x \longrightarrow \infty} \frac{1/x}{1} = 0.$$

If $a_n = n \cdot \sin(1/n)$, then

$$\lim_{n \longrightarrow \infty} a_n = \lim_{x \longrightarrow \infty} x \sin(1/x) = \lim_{x \longrightarrow 0} \frac{\sin x}{x} = \lim_{x \longrightarrow 0} \frac{(\sin x)'}{x'} = \lim_{x \longrightarrow 0} \frac{\cos x}{1} = 1.$$

This approach is *not* suitable for many sequences, including those involving $n!$ and $(-1)^n$.

(5) If a sequence $\{a_n\}$ is defined recursively as $a_{n+1} = f(a_n)$, for some function f and with some initial condition, and it converges to a, then $a = f(a)$; this is obtained by taking the limit of both sides of $a_{n+1} = f(a_n)$. So *if* the sequence a_n is known to have a limit, one simply needs to solve the equation $a = f(a)$; it may have several solutions, but it should be possible to rule out all but one of them as possible limits (perhaps only one solution of $a = f(a)$ is non-negative and $a_n > 0$ for all n). This trick applies in

8.1 Example 12: $a_1 = 2, \ a_{n+1} = \dfrac{a_n + 6}{2}$

8.1 #40: $a_1 = \sqrt{2}, \ a_{n+1} = \sqrt{2a_n}$, $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$

8.1 #46: $a_1 = \sqrt{2}, \ a_{n+1} = \sqrt{2 + a_n}$, $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

Note: given the right-most presentations of sequences on the second and third lines above, you should be able to convert them to the recursive definitions in the middle of the two lines.

(6) Before applying the trick in (5), one has to know that the sequence $\{a_n\}$ has a limit at all. The convergence/divergence test for sequences which is suitable for all three examples in (5) is the Monotonic Sequence Theorem:

> if $a_n \leq a_{n+1}$ and $a_n \leq M$ for all n ($\geq$ some N), then $\{a_n\}$ converges and $\lim_{n \longrightarrow \infty} a_n \leq M$
> if $a_n \geq a_{n+1}$ and $a_n \geq m$ for all n ($\geq$ some N), then $\{a_n\}$ converges and $\lim_{n \longrightarrow \infty} a_n \geq m$

In the first case, the sequence is increasing with n and is climbing below some "roof" M; as it keeps climbing, but cannot escape past the roof, it must approach some level below the roof (or the roof itself). In the second case, the sequence is decreasing with n and is descending toward some "floor" m; as it keeps descending, but cannot escape past the floor, it must approach some level above the floor (or the floor itself).

(7) The second main convergence test for sequences is the *Squeeze Theorem for Sequences:*

> if $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences such that $b_n \le a_n \le c_n$ for all n ($\ge$ some N), $\{b_n\}$ and $\{c_n\}$ converge, and $\lim\limits_{n\longrightarrow\infty} b_n = \lim\limits_{n\longrightarrow\infty} c_n$, then $\{a_n\}$ also converges and
>
> $$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} c_n$$

You might apply this to a specified sequence $\{a_n\}$ and appropriately chosen simpler sequences $\{b_n\}$ and $\{c_n\}$ which converge to the *same* limit and squeeze $\{a_n\}$ in between. For example, if

$$a_n = \frac{n}{n+1} + \frac{\cos n}{n},$$

you might take $b_n = n/(n+1) - 1/n$ and $c_n = n/(n+1) + 1/n$ so that

$$b_n \le a_n \le c_n, \qquad \lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} c_n = \lim_{n\longrightarrow\infty} \frac{n/n}{n/n+1/n} = 1;$$

this implies that $\{a_n\}$ also converges and its limit is also 1. If $a_n = 7^n/n!$, then

$$a_{7+n} = \frac{7^{7+n}}{(7+n)!} = \frac{7^7}{7!} \cdot \frac{7}{8} \cdot \frac{7}{9} \cdot \ldots \frac{7}{7+n} \le a_7 \cdot \left(\frac{7}{8}\right)^n;$$

so the sequence a_{7+n} is squeezed between the constant sequence $b_n = 0$ and the geometric sequence $c_n = a_7(7/8)^n$ which converges to 0 by (8) below. This implies that the sequences $\{a_n\}$ also converges to 0. The practical use of the *Squeeze Theorem for Sequences* is rather limited though. For example, in the first case above, you know that $|\cos n| \le 1$ and thus $(\cos n)/n \longrightarrow 0$; so

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} \frac{n}{n+1} = 1.$$

The second case can be dealt with by using the Monotonic Sequence Theorem (2nd case) to conclude that the sequence converges and then applying the trick described in (5) above to the recursion $a_{n+1} = (7/(n+1))a_n$ to actually find the limit.

(8) A sequence of the form $c, cr, cr^2, cr^3, \ldots$ is called geometric. It is not difficult to determine whether it converges:

> the geometric *sequence* $c, cr, cr^2, cr^3, \ldots$ with $c \ne 0$
>
> - converges if $-1 < r \le 1$ (to 0 if $-1 < r < 1$; to 1 if $r = 1$);
> - diverges if $r \le -1$ or $r > 1$.

Note that the convergence/divergence statement for geometric *series*, (6) below, is slightly different.

(9) Finally, there are *Limit Rules for Convergent Sequences*, which are more or less as expected:

> if $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any number,
>
> $$\lim_{n\longrightarrow\infty}(a_n \pm b_n) = \lim_{n\longrightarrow\infty} a_n \pm \lim_{n\longrightarrow\infty} b_n, \qquad \lim_{n\longrightarrow\infty} ca_n = c \cdot \lim_{n\longrightarrow\infty} a_n$$
>
> $$\lim_{n\longrightarrow\infty}(a_n b_n) = \big(\lim_{n\longrightarrow\infty} a_n\big) \cdot \big(\lim_{n\longrightarrow\infty} b_n\big) \qquad \lim_{n\longrightarrow\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\longrightarrow\infty} a_n}{\lim\limits_{n\longrightarrow\infty} b_n} \quad \text{if } \lim_{n\longrightarrow\infty} b_n \ne 0$$

The second equation on the first line is a special case of the first equation on the second line: just take $b_n = c$ for all n. Note that the sequences $\{a_n \pm b_n\}$, $\{a_n b_n\}$, and $\{a_n/b_n\}$ can converge even if the sequences $\{a_n\}$ and $\{b_n\}$ do not; in such cases, the limit rules are useless. Typically the limit rules are used to compute limits of sequences; in some cases they could also be used to test for convergence. For example, *if* the sequence $\{a_n\}$ converges, then the sequence $\{a_n \pm b_n\}$ converges if and only if the sequence $\{b_n\}$ does.

Series

(1) A series (or infinite series) is the sum of all terms in a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots\,, \tag{5}$$

where $a_1, a_2, \dots$ is some sequence. The lower limit in the summation need not be 1; if a_0 is the first term of the corresponding sequence, then the lower limit in the sum is 0. Associated to the infinite sum (5) is the sequence of finite partial sums,

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \qquad s_n = \sum_{k=1}^{k=n} a_k.$$

The (infinite) series (5) is said to converge to s if the sequence $\{s_n\}$ of the *partial sums* (**not** the original sequence $\{a_n\}$!!!) converges to s; if the partial sums s_n do not have a limit, the series (5) is said to diverge. Thus, if the series (5) converges to some number s, then the partial sums s_n approach s and so

$$a_n = s_n - s_{n-1} = \big(s_n - s\big) - \big(s_{n-1} - s\big)$$

approaches 0. This gives the **most important statement** regarding convergence of series:

> if the sequence $\{a_n\}$ does not converge or it converges, but $\lim_{n\longrightarrow\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ does not converge

For example, the series $\sum_{n=1}^{\infty} (-1)^n$ and $\sum_{n=1}^{\infty} \cos(n\pi/2)$ do not converge, because neither of the sequences $\{(-1)^n\}$ and $\{\cos(n\pi/2)\}$ converges to 0 (in fact, neither of the two sequences converges at all). The partial sums s_n in the first case alternate between -1 and 0 and so indeed do not approach any number; in the second case, the partial sums cycle through $0, -1, -1, 0$ and so do not approach any number either.

WARNING: The most important statement about convergence of power series *can never be used to conclude that a series converges.* For example, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, according to the p-series theorem below, even though $1/n \longrightarrow 0$.

(2) Computing the sum of an infinite series is usually difficult, but possible in some cases. A geometric series is the sum of a geometric sequence and so has the form $\sum_{n=0}^{\infty} cr^n$. By the most important statement about convergence of sequences, this sum does not converge if $|r| \geq 1$ and $c \neq 0$. Otherwise, its sum is well-known:

$$\boxed{\begin{array}{l} \displaystyle\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r} \quad \text{if } \ |r|<1 \text{ (note the lower limit on the sum)} \\ \displaystyle\sum_{n=0}^{\infty} cr^n \quad \text{does not converge if } |r| \geq 1 \text{ and } c \neq 0 \end{array}} \tag{6}$$

As an application, we can write the number $2.1\overline{37} = 2.1373737\ldots$ as a simple fraction:

$$2.1\overline{37} = 2.1 + .037 + .037 \cdot \frac{1}{100} + .037 \cdot \frac{1}{100^2} + \ldots = \frac{21}{10} + \frac{37/1000}{1-\frac{1}{100}} = \frac{21}{10} + \frac{37/10}{99}$$
$$= \frac{21 \cdot 99 + 37}{990} = \frac{2116}{990}$$

This is another example when skipping steps might increase the chance of a computational error.

(3) Infinite series can also be summed up in the cases of telescoping cancellation, $\sum_{n=1}^{\infty} (b_n - b_{n+m})$ for some fixed integer $m \geq 0$ and sequence $\{b_n\}$ converging to 0:

$$\boxed{\sum_{n=1}^{\infty} (b_n - b_{n+m}) = \sum_{n=1}^{n=m} b_n \quad \text{if } \lim_{n \longrightarrow \infty} b_n = 0, \ m \geq 0} \tag{7}$$

This is frequently used in conjunction with partial fractions. For example,

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)-(n-1)} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$
$$= \frac{1}{2} \left(\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \ldots \right)$$
$$= \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}$$

In this case, $b_n = 1/(n-1)$ for $n \geq 2$ and $m=2$. Generally, re-writing LHS of (7) as

$$\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} b_{n+m}$$

will constitute a serious error, since these two sums may not converge. For example,

$$\sum_{n=2}^{\infty} \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge by the p-series theorem below. The condition $\lim_{n\longrightarrow\infty} b_n = 0$ in (7) is absolutely **essential**. For example, the series

$$\sum_{n=1}^{\infty} \ln\big(\frac{n+1}{n}\big) = \sum_{n=1}^{\infty}\big(\ln(n+1) - \ln n\big)$$

does not converge, because the sequence of the partial sums

$$s_n = \sum_{k=1}^{k=n}\big(\ln(k+1) - \ln k\big) = \ln(n+1) - \ln 1 = \ln(n+1)$$

does not converge. The formula (7) with $b_n = -\ln n$ and $m = 1$ would have produced $b_1 = 0$ for the sum of the infinite series, which is impossible since all terms in the sum are positive. The formula (7) cannot be applied in this case because the limit condition in (7) is not satisfied.

(4) There are many cases when it can be determined whether a series converges, but it is hard to determine its sum (this is relatively rare for sequences). There are several convergence tests for series which, unlike the most important statement for series above, can determine convergence. All of the tests in 8.3 deal with series that have only positive terms a_n (for $n \geq$ some N); series with terms of different signs are in fact more likely to converge and will be considered in 8.4 (not on the 2nd midterm). In some cases, different tests can be used to determine whether a sequence converges.

(5) The Limit Comparison Test for convergence of series states that

if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, the sequence a_n/b_n converges, and the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$

For example, suppose we want to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{4^n}$ converges. With $a_n = n/4^n$ and $b_n = 1/2^n$,

$$\lim_{n\longrightarrow\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\longrightarrow\infty}\left(\frac{n}{2^n}\right) = 0$$

and $\sum_{n=1}^{\infty} b_n$ converges by the geometric series test (6); so $\sum_{n=1}^{\infty} a_n$ also converges. The same argument applies to $\sum_{n=1}^{\infty} \frac{n^p}{r^n}$ for any $r > 1$. More typically, the Limit Comparison Test is applied to series like

$$\sum_{n=1}^{\infty}\frac{1}{2^n - n}\,, \qquad \sum_{n=2}^{\infty}\frac{1}{n^2 - n}\,, \qquad \sum_{n=1}\sin^p(1/n)\,;$$

the summands in these series "asymptotically approximate" $1/2^n$, $1/n^2$, and $1/n^p$, respectively. The Limit Comparison Test with the roles of a_n and b_n reversed leads to a divergence test for series:

if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, the sequence a_n/b_n converges, $\lim_{n\longrightarrow\infty}(a_n/b_n) \neq 0$, and the series $\sum_{n=1}^{\infty} b_n$ diverges, then so does the series $\sum_{n=1}^{\infty} a_n$

In contrast to the convergence test above, there is the extra condition that a_n/b_n not approach 0; this makes sense since otherwise we could take $a_n\!=\!0$, regardless of what b_n is.

(6) A close cousin to the Limit Comparison Test is the more evident *Comparison Test* for series:

> if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, $a_n \le b_n$ for all n ($\ge$ some N), and the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\displaystyle\sum_{n=1}^{\infty} a_n$

This test with the roles of a_n and b_n reversed leads to a divergence test for series:

> if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, $a_n \ge b_n$ for all n ($\ge$ some N), and the series $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then so does the series $\displaystyle\sum_{n=1}^{\infty} a_n$

The Limit Comparison Test follows from the Comparison Test, but is likely to be more useful in this course.

(7) A very useful convergence test for series is the Integral Test:

> if f is a continuous, positive, and decreasing function on $[1,\infty)$, then the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ does

A corollary of this test is the p-series Theorem:

> the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p\!>\!1$

(8) Finally, there are also *Rules for Convergent Series*, which are more or less as expected:

> if the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converge and c is any number,
>
> $$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n\,, \qquad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

Note that these rules do not extend to multiplication and division, unlike what is the case for sequences. The series $\displaystyle\sum_{n=1}^{\infty}(a_n \pm b_n)$ can converge even if the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ do not; in such cases, the above rules are useless. Typically these rules are used to compute sums of series; in some cases they could also be used to test for convergence. For example, *if* the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then the series $\displaystyle\sum_{n=1}^{\infty}(a_n \pm b_n)$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} b_n$ does.

Good luck on the midterm