

## MAT 127: Calculus C, Fall 2009

### Course Summary I

**Extremely Important:** what it means for a function to solve a differential equation or initial-value problem and how to check this; implicitly defined solutions; general solutions vs. specific solutions.

**Very Important:** descriptive analysis of first-order differential equations, including sketches as in Figure 1 below; direction fields and solution curves; separable equations; general structure of solutions (the number of  $C$ 's); linear approximations of functions; Euler's formula.

**Important:** finding solutions to separable first-order equations and linear homogeneous second-order equations with constant coefficients; finding solutions to initial-value problems involving such equations; Euler's method and its geometric interpretation; autonomous first-order equations; application problems (leading to separable equations).

### Terminology

(1) A *first-order differential equation* is an equation that involves a function, its derivative, and the independent variable,

$$R(x, y, y') = 0, \quad y = y(x), \quad (1)$$

and cannot be simplified, through algebraic means, to a relation  $\tilde{R}(x, y) = 0$ . In (1),  $R$  is a function of three variables. Most first-order differential equations arising in applications can be put into the form

$$y' = f(x, y), \quad y = y(x), \quad (2)$$

where  $f$  is a function of two variables. An initial-value problem, for a first-order differential equation, is a set of conditions:

$$R(x, y, y') = 0 \quad \text{or} \quad y' = f(x, y), \quad y = y(x), \quad y(x_0) = y_0. \quad (3)$$

The last condition in (3) is the *initial-value requirement* for (3).

(2) A *solution* of (1), or of (2), is a function  $y = y(x)$  that satisfies (1), or (2). In order to check if a given function  $y = y(x)$  is a solution of (1), or of (2), one must compute  $y'(x)$  and plug in  $y$  and  $y'$  into (1), or (2), to see if the equality holds. A *solution* of the initial-value problem (3) is a function  $y = y(x)$  that satisfies the differential equation *and* the initial-value requirement in (3). In order to check if a given function  $y = y(x)$  is a solution of (3), one must check whether  $y = y(x)$  is a solution of the differential equation and that  $y(x_0) = y_0$ ; the latter is usually easier to do and so should be done first. Typically, but not always, (3) will have a unique solution.

(3) A *solution curve* for the first-order differential equation (1), or for (2), is the graph, in  $xy$ -plane, of a solution  $y = y(x)$  of (1), or of (2). Typically, but not always, solution curves for the same first-order differential equation will not intersect, because typically (3) has a unique solution. A

*solution curve* for the initial-value problem (3) is the graph of a solution  $y = y(x)$  of (3). Such a graph **must** pass through the point  $(x_0, y_0)$ .

*Caution:* While the solution curves for the simplest first-order differential equations, i.e. (6) below, differ by vertical shifts, this is *not* the case for other first-order differential equations.

(4) The *direction field* for (2) is usually thought of as a diagram, in the  $xy$ -plane, consisting of short line segments of slope  $y' = f(x, y)$  through a number of points  $(x, y)$ . Since the derivative of a function  $y = y(x)$  is the slope of the tangent line to the graph of  $y$ , a solution curve for (2) is everywhere tangent to the direction field. In particular, if the direction field is drawn at sufficiently many points, one can pretty much see solution curves.

(5) A second-order linear homogeneous differential equation is an equation of the form

$$y'' + by' + cy = 0, \quad b = b(x), \quad c = c(x), \quad y = y(x). \quad (4)$$

An *initial value problem* for (4) is a set of conditions

$$y'' + by' + cy = 0, \quad y = y(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1. \quad (5)$$

(6) A *solution* of (4) is a function  $y = y(x)$  that satisfies (4). If  $y_1$  and  $y_2$  are solutions of (4), so is any linear combination  $C_1y_1 + C_2y_2$  of  $y_1$  and  $y_2$ . In order to check if a given function  $y = y(x)$  is a solution of (4), one must compute  $y'(x)$  and  $y''(x)$  and plug in  $y$ ,  $y'$ , and  $y''$  into (4) to see if the equality holds. A *solution* of the initial-value problem (5) is a function  $y = y(x)$  that satisfies the differential equation *and* the two initial-value requirements in (5). Every initial-value problem (5) has a unique solution, provided the functions  $b$  and  $c$  are continuous near  $x_0$ . In order to check if a given function  $y = y(x)$  is a solution of (5), one must check whether  $y = y(x)$  is a solution of the differential equation and that  $y(x_0) = y_0$  and  $y'(x_0) = y_1$ ; the last two requirements are usually easier to check and so they should be checked first.

*Caution:* For second-order equations, initial value problems must include an initial requirement on the derivative, e.g.  $y'(x_0) = y_1$ . Thus, the graphs of solutions of second-order ODEs intersect, but they cannot be tangent to each other.

### Finding Solutions of Some First-Order Differential Equations

(1) The simplest first-order differential equations to solve are those of the form

$$y' = f(x), \quad y = y(x). \quad (6)$$

They are solved by taking the indefinite integral of both sides:

$$\boxed{y' = f(x), \quad y = y(x) \quad \Longrightarrow \quad y = \int f(x) dx}$$

The *solution curves* of (6) differ by vertical shifts. An initial-value problem for (6) is solved by

$$\boxed{y' = f(x), \quad y(x_0) = y_0 \quad \Longrightarrow \quad y(x) = y_0 + \int_{x_0}^x f(u) du}$$

(2) *Separable* first-order differential equations are the equations of the form

$$y' = f(x) \cdot g(y), \quad y = y(x). \quad (7)$$

Equation (7) is solved by writing  $y' = \frac{dy}{dx}$ , moving all expressions involving  $y$  to LHS and all expressions involving  $x$  to RHS, and integrating both sides:

$$\boxed{\frac{dy}{dx} = f(x) \cdot g(y), \quad y = y(x) \quad \Longrightarrow \quad \frac{dy}{g(y)} = f(x)dx \quad \Longrightarrow \quad \int \frac{dy}{g(y)} = \int f(x)dx}$$

Once the two integrals are computed, one obtains a relation between  $y$  and  $x$  of the form

$$G(y) = F(x) + C \quad \Longleftrightarrow \quad G(y) - F(x) = C. \quad (8)$$

These relations define solutions  $y=y(x)$  of (7) *implicitly*. In some cases, it is possible to solve (8) for  $y=y(x)$ . An initial-value problem for (7) is solved by

$$\boxed{\frac{dy}{dx} = f(x) \cdot g(y), \quad y(x_0) = y_0 \quad \Longrightarrow \quad \frac{dy}{g(y)} = f(x)dx \quad \Longrightarrow \quad \int_{y_0}^y \frac{dz}{g(z)} = \int_{x_0}^x f(u)du}$$

Alternatively, one can first find the general solution and then find the constant  $C$  by plugging in the initial conditions. It is the easiest to find  $C$  as soon as it appears, i.e. plug in the initial conditions into (8) and then solve for  $y=y(x)$ , instead of first solving (8) for  $y=y(x)$  and then solving for  $C$ .

*Caution:* (i) This separation-of-variables method involves division by  $g=g(y)$  and may miss some of the constant solutions of (7). Such solutions are necessarily of the form  $y=y^*$ , where  $y^*$  is a real number such that  $g(y^*)=0$ .

(ii) If you are solving an initial-value problem and it is possible to solve for  $y=y(x)$  explicitly, make sure you take the correct branch, if there is more than one, of the appropriate level curve of  $H=F-G$ , e.g. the positive or negative square root, and not both. The correct branch is the one satisfying the initial condition  $y(x_0)=y_0$ .

### Finding Solutions of Some Second-Order Differential Equations

The general solution of a *second-order linear homogeneous* equation with constant coefficients

$$y'' + by' + cy = 0, \quad b, c = \text{const}, \quad y = y(x), \quad (9)$$

is determined by the two roots,  $r_1$  and  $r_2$ , of the *associated polynomial*

$$r^2 + br + c = 0. \quad (10)$$

The general solution can be of two or three different forms, depending on whether one is looking for complex or real solutions:

$$\boxed{y'' + by' + cy = 0, \quad y = y(x) \quad \Longrightarrow \quad y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x} \quad \text{if } r_1 = r_2 \quad \Longleftrightarrow \quad b^2 = 4c}$$

$$\boxed{y'' + by' + cy = 0, \quad y = y(x) \quad \Longrightarrow \quad y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad \text{if } r_1 \neq r_2 \quad \Longleftrightarrow \quad b^2 \neq 4c}$$

If the coefficients  $b$  and  $c$  are real, the roots  $r_1$  and  $r_2$  of (10) are either real or complex conjugates of each other. In the latter case, Euler's formula,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

can be used to extract the general real solution from the general complex solution:

$$y'' + by' + cy = 0 \implies y(x) = C_1 e^{px} \cos qx + C_2 e^{px} \sin qx, \quad p = -\frac{1}{2}b, \quad q = \frac{1}{2}\sqrt{4c - b^2}, \quad \text{if } b^2 < 4c$$

The numbers  $p$  and  $q$  are related to the roots  $r_1$  and  $r_2$  by  $r_1, r_2 = p \pm iq$ .

### Euler's Method

(1) Euler's method is used to estimate the value  $y(b)$  of the solution  $y = y(x)$  to a *first-order* initial-value problem

$$y' = f(x, y), \quad y(a) = y_0, \tag{11}$$

at  $b$ , for  $b > a$ . This is especially useful when the explicit solution  $y = y(x)$  of (11) cannot be found. Euler's method can also be used if  $f$  is known only at a discrete grid of points  $(x_i, y_j)$ , as the case may well be in an experimental setting.

(2) Euler's method is a *fixed-step* method. This means that we break up the interval  $[a, b]$  into  $N$  segments  $[x_i, x_{i+1}]$  of equal length  $h = (b - a)/N$ , i.e.

$$x_0 = a, \quad x_1 = x_0 + h = a + h, \quad \dots \quad x_{N-1} = x_{N-2} + h = a + (N-1)h, \quad x_N = x_{N-1} + h = a + Nh = b.$$

We then give an estimate  $y_i$  for the value of the function  $y$  at  $x_i$ . More precisely, we give an estimate  $y_1$  for  $y(x_1)$ , where  $y = y(x)$  is the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

We then use the same procedure to give an estimate  $y_2$  for  $\tilde{y}_1(x_2)$ , where  $\tilde{y}_1 = \tilde{y}_1(x)$  is the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_1) = y_1.$$

Since  $y_1$  is an estimate for  $y(x_1)$ ,  $y_2$  will also be an estimate for  $y(x_2)$ . At the  $i$ -th step of this construction, we give an estimate  $y_{i+1}$  for  $\tilde{y}_i(x_{i+1})$ , where  $\tilde{y}_i = \tilde{y}_i(x)$  is the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_i) = y_i.$$

After  $N$  steps, we end up with an estimate  $y_N$  for  $\tilde{y}_{N-1}(x_N)$ , where  $\tilde{y}_{N-1} = \tilde{y}_{N-1}(x)$  is the solution to the initial-value problem

$$y' = f(x, y), \quad y(x_{N-1}) = y_{N-1}.$$

This number  $y_N$  will also be an estimate for  $y(b)$ .

(3) In Euler's method, we take

$$y_{i+1} = y_i + s_i h, \quad \text{where} \quad s_i = f(x_i, y_i).$$

Since  $\tilde{y}_i(x_i) = y_i$  and  $\tilde{y}'_i(x_i) = f(x_i, y_i) = s_i$ ,  $y_{i+1}$  estimates  $\tilde{y}_i(x_{i+1})$  by a linear approximation in  $h$ . This means that instead of moving from  $x_i$  to  $x_{i+1}$  along the graph of  $\tilde{y}_i$  (which we do not), we move along the tangent line to this graph at  $(x_i, y_i)$ ; we know what the tangent line is, since it is determined by the condition that it passes through  $(x_i, y_i)$  and its slope is  $s_i$ . This computation can be carried out using a five-column table, as done in the solutions to PS2. The columns are labeled  $i$ ,  $x_i$ ,  $y_i$ ,  $s_i = f(x_i, y_i)$ , and  $y_{i+1} = y_i + s_i h$ , with  $f(x_i, y_i)$  replaced by the expression from (11) and  $h$  by the actual step size (a small number). The rows in the first column are then filled out by the integers from 0 to  $n - 1$ . The corresponding entries in the second column are filled with the numbers  $x_0, x_1, \dots, x_{N-1}$ ; these numbers begin with the given  $x_0 = a$ , increase by  $h$  from row to row, and end with  $x_{N-1} = b - h$ . The third entry in the  $i = 0$  row is the given initial value  $y_0$ . From here one computes to the right as indicated by the column labels and copies the last entry in each to the third column of the following row. The entry in the bottom right corner is our estimate  $y_N$  for  $y(b)$ .

### Applications

(1) Find an equation for the curve with slope  $f(x, y)$  at  $(x, y)$  and passing through a point  $(x_0, y_0)$ . Since the slope of the graph of  $y = y(x)$  at  $(x, y(x))$  is  $y'(x)$ , this reduces to solving the initial-value problem

$$y' = f(x, y), \quad y = y(x), \quad y(x_0) = y_0;$$

the last condition means that the graph of  $y$  passes through  $(x_0, y_0)$ . You should simplify your final answer as much as possible, without necessarily finding an explicit expression for  $y = y(x)$  since you are asked for an equation of a curve, not of a function. For example,  $x^2 + y^2 = 1$  is a nicer expression for a curve, then  $y = \pm\sqrt{1 - x^2}$ .

(2) Find orthogonal trajectories to a family of curves  $H(k, x, y) = 0$ . In this case, for each fixed  $k$  the equation  $H(k, x, y) = 0$  describes a curve in the  $xy$ -plane. Using chain rule, we differentiate this equation with respect to  $x$ , which should lead to an expression like  $y' = F(k, x, y)$ . If this expression involves  $k$ , it should be possible to eliminate  $k$  using  $H(k, x, y) = 0$ . One then obtains a differential equation of the form  $y' = f(x, y)$ ; it describes the original family curves, not the orthogonal curves. For the latter, the slope is the negative reciprocal, so the problem reduces to finding the general solution to the differential equation

$$y' = -\frac{1}{f(x, y)}.$$

Once this is done, representatives of the two families of curves should intersect at right angles if properly sketched.

(3) *Mixing problems.* In these problems the independent variable is usually  $t$  (time), measured in some units, and  $y = y(t)$  is the amount of some substance in a thoroughly mixed solution/mixture

contained in some “reservoir” (water tank, room, etc.). The mixture might be leaving the reservoir at some fixed rate, while another mixture (or several other mixtures) might be entering the reservoir. The main difficulty in such problems is to set up a differential equation for  $y(t)$ . It will be of the form

$$y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t), \quad y = y(t).$$

The outgoing flow rate of the substance,  $y'_{\text{out}}(t)$ , is the concentration of the substance in the mixture times the outgoing flow rate of the mixture; while the outgoing flow rate of the mixture is typically fixed, the concentration typically changes with time. The incoming flow rate of the substance,  $y'_{\text{in}}(t)$ , is the concentration of the substance in the incoming mixture times the incoming flow rate of the mixture (this might have to be summed up over several incoming mixtures); this number is typically constant. So you should end up with an initial-value problem of the form

$$y' = a - by, \quad y = y(t), \quad y(0) = y_0,$$

where  $y_0$  is the initial amount of the substance in the reservoir and  $a, b > 0$ ; both are constants if the volume in the reservoir is kept constant (otherwise, only  $a$  is constant). If  $a, b$  are constants, the above equation is separable and so can be solved. If you do this correctly, the concentration of the substance in the reservoir should approach the (weighted) concentration of the substance in the incoming mixture(s). *Make sure your final answer has the correct physical units.*

(4) *Exponential growth/decay equation.* In this case, the independent variable is usually  $t$  (time), measured in some units, and the main dependent variable  $y(t)$  is described

$$y(t) = y(0)e^{rt}$$

which satisfies the differential equation  $y' = ry$ ,  $y = y(t)$ . The first equation is the equation you should start with and then try to figure out what the numbers  $y(0)$  and  $r$  are. These might be given in the statement of the problem or you may have to find them knowing that  $y(t)$  is given by the above formula for some  $y(0)$  and  $r$ . The function  $y(t)$  will typically be the size of a population, amount of a radioactive substance, difference with the ambient temperature, or bank/loan balance at time  $t$  measured in certain units (which you may need to specify). In all of these cases  $y(t) > 0$  for all  $t$ ; in the middle two cases  $r < 0$ , while in the other two cases  $r > 0$ . *Make sure your final answer has the correct physical units.*

(5) *Logistic growth equation.* In this case, the independent variable is usually  $t$  (time), measured in some units, and the main dependent variable  $y(t)$  is normally the size of a population at time  $t$  growing with limited resources. You should begin any such problem with the formula

$$y(t) = \frac{K}{1 - \left(\frac{y(0)-K}{y(0)}\right)e^{-rt}}$$

which solves the differential equation

$$y' = ry \left(1 - \frac{y}{K}\right), \quad y = y(t).$$

If you forget the first formula, you can recover it by solving the differential equation. Since the equation is separable, this is doable, though requires some effort. In this setting,  $K > 0$  represents the carrying capacity of the available resources, measured in whatever units the population  $y(t)$  is measured;  $y(t)$  approaches  $K$  as  $t \rightarrow \infty$ . Furthermore,  $r > 0$ . You may be given the numbers  $y(0)$ ,  $r$ , and  $K$  in the statement of the problem or you may have to find (some of) them based on the information provided. *Make sure your final answer has the correct physical units.*

### Autonomous First-Order ODEs

(1) An *autonomous* first-order ODE is an ODE of the form

$$y' = g(y), \quad y = y(t). \quad (12)$$

Such equations often model natural phenomena, expressing the fact these are governed by the same principles whether they start today or tomorrow. The equations in applications (3)-(5) above are all autonomous (assuming  $a, b$  are constant in (3)). Equation (12) is separable and we can solve it implicitly for  $y=y(t)$  as  $G(y)=t+C$ . However, a lot of descriptive information about (12) can be obtained without solving it.

(2) Since RHS of (12) does not involve  $t$ , the direction field of (12) does not change under horizontal shifts. Thus, a *horizontal* shift of a solution curve is again a solution curve. Furthermore, if  $y^*$  is a real number such that  $g(y^*)=0$ , the constant function  $y(t)=y^*$  is a solution of (12). Such a number  $y^*$  is an *equilibrium point* for (12) and  $y(t)=y^*$  is an *equilibrium solution* of (12). The corresponding solution curve is the horizontal line  $y = y^*$  in  $(t, y)$ -plane. The horizontal graphs of the equilibrium solutions of (12) partition the  $(t, y)$ -plane into horizontal bands  $y_1^* < y < y_2^*$ . In each band, the function  $g(y)$  does not change sign. Thus, in each single band, all solution curves of (12) either descend and approach the line  $y=y_1^*$  or ascend and approach the line  $y=y_2^*$  as  $t$  approaches  $\infty$ .

(2) Here is an example. The equilibrium solutions of

$$y' = (y + 3)^2(y + 1)(y - 3), \quad y = y(t), \quad (13)$$

are  $y = -3$ ,  $y = -1$ , and  $y = 3$ . The graphs of these solutions are the horizontal lines  $y = -3$ ,  $y = -1$ , and  $y = 3$ , shown in the third plot in Figure 1. These lines partition the  $ty$ -plane into horizontal bands  $y_1^* < y < y_2^*$ . Since solution curves of the differential equation (13) do not intersect, no solution curve can cross the graphs of the equilibrium solutions. For example, if  $y = y(t)$  is a solution of (13) such that  $y(t_0) \in (-1, -3)$  for some  $t_0$ , then  $y(t) \in (-1, -3)$  for all  $t$ . In each band, the function  $g(y)$  does not change sign. Thus, in each single band, all solution curves of (13) either descend or ascend. Furthermore, each solution curve must approach either an equilibrium solution curve or  $\pm\infty$  as  $t \rightarrow \pm\infty$ . The best way to tell whether the solution curves in the given band descend or ascend is by sketching the graph of the function  $g = g(y)$ , as done for the differential equation (13) in the first plot in Figure 1. Note that the  $y$ -intercepts of this graph correspond to the equilibrium solutions of the differential equation. The phase line in the middle of Figure 1 shows the equilibrium points for the differential equation (13), or the  $y$ -intercepts of the graph of  $g$ .

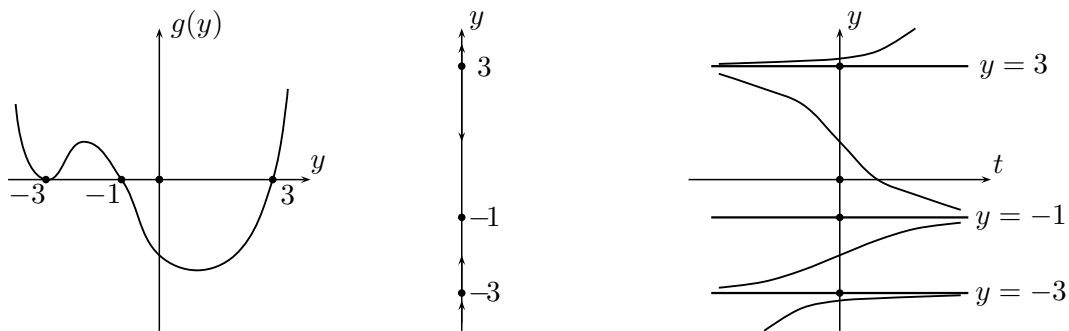


Figure 1: Plots for the differential equation  $y' = g(y) = (y + 3)^2(y + 1)(y - 3)$

It also indicates, using arrows, whether the solution curves in each band cut out by the horizontal equilibrium-solution lines ascend and descend. The arrow corresponding to a segment of the phase line points up (down) if  $g(y)$  is positive (negative) on the this segment.

### Euler's Formula and its Implications

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Here are some consequences:

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

The double-angle formulas follow from Euler's formula:

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \quad \text{and} \quad \sin 2\theta = 2 \cos \theta \cdot \sin \theta,$$

as do the more general formulas:

$$\cos(\alpha \pm \beta) = \cos \alpha \cdot \cos \beta \mp \sin \alpha \cdot \sin \beta \quad \text{and} \quad \sin(\alpha \pm \beta) = \sin \alpha \cdot \cos \beta \pm \cos \alpha \cdot \sin \beta.$$

*Please derive all these from Euler's formula.*