CONJUGACIES BETWEEN HORSESHOE BRAIDS

ANDRÉ DE CARVALHO AND TOBY HALL

Abstract. A proof is given of part of a conjecture about the way in which the set of horseshoe braid types is organized by the forcing partial order. Horseshoe periodic orbits are labelled by their *height* (a rational number) and *decoration* (a word in the symbols 0 and 1), and it is shown that, with the exception of so-called *limit* cases, periodic orbits of the same height and decoration have the same braid type.

1. Introduction

This paper contains a proof of part of a conjecture about the way in which horseshoe braid types are organized by the forcing partial order. The notion of braid type has a natural definition in terms of the dynamical process of following periodic orbits through families of maps: period n orbits P and Q of homeomorphisms $f, g\colon D^2 \to D^2$ are said to have the same braid type if there is a continuously varying family $\{f_t\}$ of homeomorphisms from f to g, and a path P_t in $(D^2)^n$ from P to Q, such that P_t is a least period n orbit of f_t for all t. This relation is an equivalence relation on the set of all pairs (P, f), and the braid type of the periodic orbit P of f is the equivalence class containing (P, f). A braid type β is said to force a braid type γ if every homeomorphism which has a periodic orbit of braid type β also has one of braid type γ.

Forcing is a partial order on the set of all braid types, but little is known about the structure of this order. The conjecture [dCH] describes the restriction of the order to the set of braid types which arise in Smale's horseshoe map. It labels horseshoe periodic orbits by their height (a rational number) and decoration (a word in the symbols 0 and 1), and claims (amongst other things) that the family of all horseshoe braid types of given decoration is totally ordered by the forcing order, the braid types within each family being ordered inversely to their heights. In order for this statement to make sense, it is necessary that periodic orbits of the same height and decoration should have the same braid type: this is what is proved here (at least for non-limit periodic orbits, i.e. those which are not minimum elements of their family). While other parts of the conjecture have been proved in special cases, this is the first part to be proved in general, demonstrating that it is meaningful to describe horseshoe periodic orbits by height and decoration.

Date: August 2002.

Because braid types of period n orbits can be labelled by conjugacy classes in Artin's braid group B_n modulo its centre, this result translates to one about conjugacies between horseshoe braids (negative permutation braids of cyclic unimodal permutations). In this language, which is the one primarily adopted here, the result states that non-limit horseshoe braids of the same height and decoration are conjugate.

The necessary preliminaries, including the definitions of height and decoration and some results about them from earlier papers, are discussed in Section 2. Given these results, the proof of the main theorem, contained in Section 3, is quite short. The main technique used, that of repeated reversal of horseshoe periodic orbit codes, is well established, and the paper ends with a discussion of the relationship between the main theorem and an earlier result due to Holmes [Hol89].

2. Height and decoration of horseshoe braids

This section contains a brief summary of definitions and results from other papers. First, some standard facts about Smale's horseshoe diffeomorphism [Sma67] are listed, mainly in order to fix notation (see [Dev89] for a fuller account using the same notation). The description of horseshoe periodic orbits in terms of height and decoration is then discussed. The definitions and results cited have appeared in several of the authors' other papers: [dCH, Hal94] contain more details and motivation.

The standard model of the horseshoe $F\colon D^2 \to D^2$ depicted in Figure 1 is used (the dot on the square and its image indicate the orientation of the image). Symbolic dynamics (with symbols as indicated in the figure) is introduced in the usual way, yielding an itinerary homeomorphism $k\colon \Lambda \to \{0,1\}^{\mathbb{Z}}$, where Λ is the Cantor set of points with orbits lying entirely in the central square S of the horseshoe. Thus k conjugates $F|_\Lambda$ to the shift map $\sigma\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$.

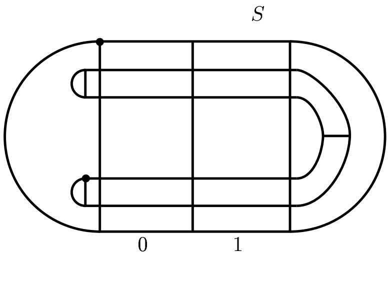

Figure 1. Smale's horseshoe

If P is a period n orbit of F, then its *code* $c_P \in \{0,1\}^n$ is defined to be the first n symbols of the itinerary $k(p)$ of its rightmost point p: thus $k(p) = \overline{c_P}$, where the overline indicates infinite repetition both to the left and the right (i.e. $\overline{w} = \dots www \cdot www \dots$, the $\cdot$ appearing before the origin of the itinerary). The choice of p as the rightmost point of P means that a word $w \in \{0,1\}^n$ is the code of some period n horseshoe orbit P if and only if it is *maximal*, i.e. w^∞ is strictly greater than $\sigma^j(w^\infty)$ in the unimodal order for $1 \leq j < n$ (where the superfix ∞ indicates infinite repetition to the right, i.e. $w^\infty = wwww \ldots \in \{0,1\}^{\mathbb{N}}$).

A period n horseshoe orbit P also gives rise naturally to an n-braid β_P. Let A_n be a standard set of n points on the horizontal diameter of D^2. Since vertical arcs in S are contracted by F the points of P have distinct horizontal coordinates, and hence there is a unique isotopy class of homeomorphisms $(D^2, P) \to (D^2, A_n)$ containing an element H which, as an automorphism of D^2, is isotopic to the identity by an isotopy $\{H_t\}$ with the property that each H_t maps vertical arcs in D^2 to vertical arcs in D^2. Conjugating $F\colon (D^2, P) \to (D^2, P)$ by such an H yields a well-defined isotopy class of homeomorphisms $(D^2, A_n) \to (D^2, A_n)$, and hence, using the standard identification of the mapping class group of (D^2, A_n) with $B_n/Z(B_n)$, a well-defined element γ_P of $B_n/Z(B_n)$ (recall that the centre $Z(B_n)$ of the braid group B_n is the infinite cyclic subgroup generated by a full twist). Then $\beta_P \in B_n$ is the element of γ_P with negative exponent sum closest to zero. A braid $\beta \in B_n$ is equal to β_P for some P if and only if it is a negative permutation braid of a cyclic unimodal permutation $\pi \in S_n$: this permutation π is precisely the one induced by F on the horizontal coordinates of the points of P.

Two horseshoe period n orbits P and Q have the same *braid type* [Boy84] if γ_P and γ_Q are conjugate elements of $B_n/Z(B_n)$, or equivalently (because of the exponent sum condition) if β_P and β_Q are conjugate in B_n. This happens if and only if there is an isotopy $\{F_t\}\colon F \simeq F$ from the horseshoe to itself and a path P_t in $(D^2)^n$ with $P_0 = P$ and $P_1 = Q$, with the property that P_t is a (least) period n orbit of F_t for all t (i.e. P can be followed through a family of homeomorphisms and end up as Q). In studying horseshoe-creating bifurcations, it is therefore important to understand the conjugacy relation between horseshoe braids. The following definitions will be useful:

Definitions 1. Write $\mathcal{H}_n \subseteq B_n$ for the set of horseshoe n-braids, and $\mathcal{H}$ for the disjoint union of the $\mathcal{H}_n$ $(n \geq 1)$. Given $\beta \in \mathcal{H}_n$, denote by $n(\beta)$ the number of elements of $\mathcal{H}_n$ which are conjugate (in B_n) to β. If $n(\beta) = 1$, then β is said to be *lone*.

It is well known that if P and Q are distinct period n horseshoe orbits, then $\beta_P = \beta_Q$ if and only if c_P and c_Q differ only in their final symbol. In particular, a horseshoe n-braid $\beta \in \mathcal{H}_n$ can be described uniquely by a word $c(\beta) \in \{0,1\}^{n-1}$, namely the first $n-1$ symbols of c_P, where P is a period n horseshoe orbit with $\beta_P = \beta$.

Definition 2. A word $w \in \{0,1\}^{n-1}$ is a *braid code* if either $w0$ or $w1$ is maximal.

Thus w is a braid code if and only if there is a horseshoe n-braid β with $c(\beta) = w$: in this case w is called the braid code *of* β.

Remark 1. For readers used to working with the codes of horseshoe periodic orbits, it is important to notice that the braid code of a horseshoe braid is missing the final symbol of the code of the corresponding periodic orbit or orbits.

The content of Theorem 6 below is that, for 'most' horsehoe periodic orbits P, there is another symbol of c_P which can be changed to yield a horseshoe braid conjugate to β_P. In order to describe which symbol this is, it is necessary to define the *height* of a horseshoe periodic orbit or, equivalently, of a horseshoe braid. The first step is to describe two classes of horseshoe braids which are known to be lone, namely the *finite order* and *NBT* braids. A consequence of Theorem 6 is that if $\beta \in \mathcal{H}$ is lone, then it is either finite order, NBT, or a *limit braid*. In the following definitions, and throughout the remainder of the paper, all rationals m/n are assumed to be written in reduced form. The definitions are given first in a readily accessible form, with a more practical equivalent formulation presented afterwards.

Definitions 3. Given $q = m/n \in \mathbb{Q} \cap (0, 1/2]$, define words $c_q \in \{0,1\}^{n+1}$ and $d_q \in \{0,1\}^{n-1}$ as follows. Let L_q be the straight line in $\mathbb{R}^2$ from $(0,0)$ to (n,m). For $0 \le i \le n$, let $s_i = 1$ if L_q crosses some line $y =$ integer for $x \in (i-1, i+1)$, and $s_i = 0$ otherwise. Then $c_q = s_0 s_1 \dots s_n$ and $d_q = s_0 s_1 \dots s_{n-2}$. A horseshoe braid β is *finite order* if $c(\beta) = d_q$ for some q, and is *NBT* if $c(\beta) = c_q$ for some q.

It is known [Hal94] that d_q and c_q are braid codes for all $q \in \mathbb{Q} \cap (0, 1/2]$, so that there is exactly one finite order braid α_q and one NBT braid ν_q for each such rational q. The finite order braid $\alpha_{m/n}$ is the unique element of $\mathcal{H}_n$ with the propery that the isotopy class of homeomorphisms $(D^2, A_n) \to (D^2, A_n)$ which it defines (as described above) contains an element conjugate to a rotation through angle $2\pi m/n$ [HW85].

Example 1. Figure 2 shows that $c_{3/10} = 10011011001$. Hence $\alpha_{3/10}$ has braid code 100110110, and $\nu_{3/10}$ has braid code 10011011001.

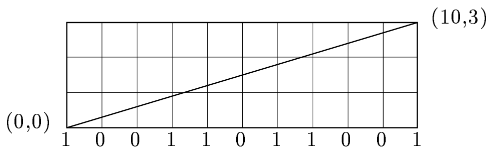

FIGURE 2. The braid code of an NBT braid

Note that the words c_q are palindromic. Their general form is indicated by the examples in Table 1, in which the column headings and row headings denote the numerator and denominator of q respectively. The $n-2m+1$ zeros are partitioned 'as even-handedly as possible' into m subwords (possibly empty), separated by 11.

	1	2	3	4
3	1001			
4	10001			
5	100001	101101		
6	1000001			
7	10000001	10011001	10111101	
8	100000001		101101101	
9	1000000001	1000110001		1011111101
10	10000000001		10011011001	
11	100000000001	100001100001	100110011001	101101101101

TABLE 1. Examples of the words c_q

The following more practical method for determining c_q is easily shown to be equivalent: given $q = m/n$, define integers $\kappa_i(q)$ for $1 \leq i \leq m$ by

$$\begin{aligned}\kappa_1(q) &= \left\lfloor \frac{n}{m} \right\rfloor - 1,\\ \kappa_i(q) &= \left\lfloor \frac{in}{m} \right\rfloor - \left\lfloor \frac{(i-1)n}{m} \right\rfloor - 2 \qquad (2 \leq i \leq m).\end{aligned}$$

Then $c_q = 10^{\kappa_1(q)}110^{\kappa_2(q)}11\ldots110^{\kappa_m(q)}1$.

Example 2. Let $q = 3/7$, so $\kappa_1(q) = \lfloor 7/3 \rfloor - 1 = 1$, $\kappa_2(q) = \lfloor 14/3 \rfloor - \lfloor 7/3 \rfloor - 2 = 4 - 2 - 2 = 0$, and $\kappa_3(q) = \lfloor 21/3 \rfloor - \lfloor 14/3 \rfloor - 2 = 7 - 4 - 2 = 1$. Thus $c_{3/7} = 10111101$ (and $d_{3/7} = 101111$).

Height is a conjugacy invariant of horseshoe braids taking rational values in $(0, 1/2]$. The next lemma [Hal94] motivates the definition, implying in particular that the height function $q\colon \{0,1\}^{\mathbb{N}} \to [0, 1/2]$ is decreasing (with respect to the unimodal order $\preceq$ on $\{0,1\}^{\mathbb{N}}$).

Lemma 1. *Let q and r be elements of $\mathbb{Q} \cap (0, 1/2]$ with $q > r$. Then $(c_q0)^\infty \prec (c_r0)^\infty$.*

Thus the elements of $\{(c_q0)^\infty : q \in (0, 1/2]\}$ are ordered by the unimodal order inversely to the parameter q. The height $q(c)$ of $c \in \{0,1\}^{\mathbb{N}}$ indicates where it fits into this chain:

Definitions 4. Let $c \in \{0,1\}^{\mathbb{N}}$. Then the *height* $q(c) \in [0, 1/2]$ of c is given by

$$q(c) = \inf\{q \in (0, 1/2] \cap \mathbb{Q} : q = 1/2 \text{ or } (c_q0)^\infty \prec c\}.$$

The *height* $q(\beta)$ of a horseshoe braid $\beta \in \bigcup_{n\geq 2} \mathcal{H}_n$ is given by

$$q(\beta) = q\left((c(\beta)0)^\infty\right).$$

If $\beta \in \mathcal{H}_2$ then $c(\beta) = 1$, and hence $q(\beta) = q((10)^\infty)$. Otherwise, $c(\beta) = 10\ldots$, and so $(c(\beta)0)^\infty$ contains the word 010. Thus the following algorithm [Hal94] can be used to calculate $c(\beta)$, and in particular the height of a horseshoe braid is rational and strictly positive.

Lemma 2. *Let $c \in \{0,1\}^{\mathbb{N}}$, and suppose that c contains the word 010. Then $q(c)$ can be calculated as follows. If c does not begin 10, then $q(c) = 1/2$. Otherwise, write*

$$c = 10^{\kappa_1}1^{\mu_1}0^{\kappa_2}1^{\mu_2}\ldots,$$

where each $\kappa_i \geq 0$, each μ_i is either 1 or 2, and $\mu_i = 1$ only if $\kappa_{i+1} > 0$ (thus κ_i and μ_i are uniquely determined by c). For each $r \geq 1$, define

$$I_r(c) = \left(\frac{r}{2r + \sum_{i=1}^r \kappa_i}, \frac{r}{(2r-1) + \sum_{i=1}^r \kappa_i}\right],$$

and let $s \geq 1$ be the least integer such that either $\mu_s = 1$ or $\bigcap_{i=1}^{s+1} I_i(c) = \emptyset$. Write $\bigcap_{i=1}^{s} I_i(c) = (x, y]$. Then

$$q(c) = \begin{cases} x & \text{if } \mu_s = 2 \text{ and } w < z \text{ for all } w \in I_{s+1}(c) \text{ and } z \in \bigcap_{i=1}^s I_i(c) \\ y & \text{if } \mu_s = 1, \text{ or } \mu_s = 2 \text{ and } w > z \text{ for all } w \in I_{s+1}(c) \text{ and } z \in \bigcap_{i=1}^s I_i(c). \end{cases}$$

Notice that some μ_i is equal to 1 (since c contains the word 010), and hence the algorithm terminates.

Example 3. Let β be the horseshoe braid with braid code 1011110111. Then $q(\beta) = q((10111101110)^\infty)$. Using the notation of the definition, $\kappa_1 = 1$, $\mu_1 = 2$, $\kappa_2 = 0$, $\mu_2 = 2$, $\kappa_3 = 1$, $\mu_3 = 2$, $\kappa_4 = 0$, and $\mu_4 = 1$. Hence $I_1 = (1/3, 1/2]$, $I_2 = (2/5, 2/4]$ (so $\bigcap_{i=1}^2 I_i = (2/5, 1/2]$), $I_3 = (3/8, 3/7]$ (so $\bigcap_{i=1}^3 I_i = (2/5, 3/7]$), and $I_4 = (4/10, 5/9]$ (so $\bigcap_{i=1}^4 I_i = (2/5, 3/7]$). Since $\mu_4 = 1$, it follows that $q(\beta) = 3/7$. Although tedious to carry out by hand, the height algorithm is easily automated [Hal].

The following result [Hal94] describes the properties of height which will be used here:

Lemma 3. *a) The height of a horseshoe braid is a conjugacy invariant.*
b) Let $\beta \in \mathcal{H}_k$ have height $q(\beta) = m/n$. Then either $\beta = \alpha_{m/n}$, or $\beta = \nu_{m/n}$, or $k > n+2$ and $c(\beta)$ has $c_{m/n}$ as an initial subword.

Corollary 4. *Finite order and NBT braids are lone.*

Proof. $\alpha_{m/n}$ (resp. $\nu_{m/n}$) is the only element of $\mathcal{H}_n$ (resp. $\mathcal{H}_{n+2}$) of height m/n. □

Definition 5. Suppose $\beta \in \mathcal{H}_k$ is neither finite order nor NBT. Let $q(\beta) = m/n$. Then the *decoration* $w \in \{0,1\}^{k-n-3}$ of β is the final length $k-n-3$ subword of $c(\beta)$.

By Lemma 3 b), if β has decoration w, then $c(\beta) = c_{q(\beta)}{}^{0}_{1}w$ (the notation ${}^{0}_{1}$ is used in the same way as the notation $\pm$: thus here it means that either $c(\beta) = c_{q(\beta)}0w$ or $c(\beta) = c_{q(\beta)}1w$). Hence height and decoration together provide an alternative way of describing horseshoe braids: given $q \in (0, 1/2] \cap \mathbb{Q}$ and $w \in \{0,1\}^n$ for some n, there are at most two horseshoe braids of height q and decoration w, namely those with braid codes $c_q{}^0_1w$ (if these words are indeed braid codes); and therefore there are at most four horseshoe periodic orbits of height q and decoration w, namely those whose codes are $c_q{}^0_1w^0_1$. Conversely, any horseshoe braid which is neither finite order nor NBT has well-defined height and decoration.

The main result of this paper, Theorem 6, says that if $c_q{}^0_1w$ are both braid codes, then the corresponding horseshoe braids are conjugate.

The following result [dCH] completes the classification of (non-finite order and non-NBT) horseshoe braids by height and decoration. The final statement is not part of the classification, but is needed in Section 3.

Lemma 5. *Given $w \in \{0,1\}^k$ for some $k \geq 0$, let $q_w \in (0,1/2] \cap \mathbb{Q}$ be given by*

$$q_w = \min_{0 \leq i \leq k+2} q\left(\sigma^i((10w0)^\infty)\right).$$

If $0 < q < q_w$ then both of the words $c_q{}^0_1w$ are braid codes of height q braids, while if $q_w < q \leq 1/2$ then neither is the braid code of a height q braid. If $q = q_w$ then neither, one, or both of the words $c_q{}^0_1w$ may be braid codes of height q braids.

Moreover, if $q < q_w$ then all four of the words $c_q{}^0_1w^0_1$ are maximal.

Remark 2. The point of the qualification 'height q' in this lemma is that it is possible, when $q > q_w$, for $c_q{}^0_1w$ to be a braid code, but in this case the corresponding horseshoe braid β does not have height q. For example, let $w = \emptyset$, the empty word, with $q(w) = 1/3$ (see Examples 4 below). Then $c_{2/5}1w = 1011011$ is a braid code, since 1011011^0_1 are both maximal words. However, this is the braid code of $\alpha_{3/8}$, with height $3/8$ rather than $2/5$.

Examples 4. a) Let $w = \emptyset$ be the empty decoration. Then

$$\begin{aligned} q_w = \min_{0 \leq i \leq 2} q\left(\sigma^i((100)^\infty)\right) &= \min\left(q((100)^\infty), q((001)^\infty), q((010)^\infty)\right) \\ &= \min\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}. \end{aligned}$$

Since $1/4 < 1/3$, there are two horseshoe braids of height $1/4$ and decoration $\emptyset$, namely those with braid codes 100010 and 100011. Since $c_{1/3} = 1001$, and 10011 and 10010 are both braid codes, there are also two horseshoe braids of height $1/3$ and decoration $\emptyset$. For $q > 1/3$, there are no horseshoe braids of height q and decoration $\emptyset$.

b) Let $w = 1$. Then

$$q_w = \min_{0 \leq i \leq 2} q\left(\sigma^i((1010)^\infty)\right) = q((10)^\infty) = \frac{1}{2}.$$

Now $c_{1/2} = 101$, and $c_{1/2}11 = 10111$ is a braid code, while $c_{1/2}01 = 10101$ is not. Thus there is only one horseshoe braid of height $1/2$ and decoration 1. For all $q \in (0, 1/2) \cap \mathbb{Q}$, there are two horseshoe braids of height q and decoration 1.

c) If $w = 10011010$ then $q_w = 1/3$, and neither of the two words $c_{1/3}{}^{0}_{1}w$ is a braid code (the straightforward check is left to the reader). Hence there are no horseshoe braids of height $q \geq 1/3$ and decoration 10011010. For all $q \in (0, 1/3) \cap \mathbb{Q}$, there are two horseshoe braids of height q and decoration 10011010.

Definition 6. Let $\beta \in \mathcal{H}$ be a horseshoe braid of height q and decoration w. If $q = q_w$ then β is said to be a *limit* braid.

The next result is the main theorem of this paper, which will be proved in Section 3.

Theorem 6. *If two non-limit horseshoe braids have the same height and decoration, then they are conjugate.*

Remark 3. Since finite order and NBT braids do not have decorations, this theorem says nothing about them, as one would expect since they are known to be lone.

Example 5. The two horseshoe braids $\beta, \gamma \in \mathcal{H}_8$ with braid codes 1001011 and 1001111 both have height $1/3$ and decoration 11. Moreover $q_{11} = 2/5$, so these are not limit braids. Hence they are conjugate. In fact, in terms of the standard Artin generators of B_8,

$$\begin{aligned} \beta &= \sigma_4^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_6^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_7^{-1}\sigma_6^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}, \\ \gamma &= \sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_6^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_7^{-1}\sigma_6^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}, \end{aligned}$$

and if $\delta = \sigma_6\sigma_2\sigma_3\sigma_4\sigma_5$ then $\delta^{-1}\beta\delta = \gamma$.

Corollary 7. *If $\beta \in \mathcal{H}$ is lone, then β is either finite order, NBT, or a limit braid.*

Remark 4. The converse of Theorem 6 is false. The smallest counterexamples are elements of $\mathcal{H}_9$: the two braids 10001010 and 10001110 of height $1/4$ and decoration 10 are conjugate to the braids 10001101 and 10001001 of height $1/4$ and decoration 01. The following conjecture is part of one discussed in [dCH].

Conjecture 8. *There is an equivalence relation on the set of decorations with the property that two horseshoe braids are conjugate if and only if they have the same height and equivalent decorations.*

Hence if this conjecture is true, then the four horseshoe braids with braid codes c_q010, c_q110, c_q001 and c_q101 are conjugate for all $q < q_{10} = q_{01} = 1/3$; and, since there are examples of rationals q such that these braids are not conjugate to the braids with height q and decoration 00 or 11, $n(\beta) = 4$ for each such braid.

The final lemma of this section, from [dCH], is an essential ingredient in the proof of Theorem 6. Given a word $w = w_0 \dots w_{k-1} \in \{0,1\}^k$ for some $k \geq 0$, denote by $\widehat{w}$ the reverse word $w_{k-1} \dots w_0$.

Lemma 9. *Let w be a decoration. Then $q_w = q_{\widehat{w}}$.*

3. Conjugate horseshoe braids

The proof of Theorem 6 is quite straightforward given the results presented in Section 2. It makes use of a well-known interpretation of the reverse $\widehat{c_P}$ of the code c_P of a horseshoe periodic orbit P (see for example [Hol89]).

Let P be a period n horseshoe orbit, and let $\widehat{P}$ be the period n orbit containing the point of itinerary $\overline{\widehat{c_P}}$. The code $c_{\widehat{P}}$ of $\widehat{P}$ is thus a cyclic permutation of $\widehat{c_P}$.

Example 6. Let P be the periodic orbit with code $c_P = 1001011$. Then $\widehat{P}$ is the periodic orbit which contains the point with itinerary $\overline{1101001}$: thus $c_{\widehat{P}} = 1001110$.

Given $\beta \in B_n$, let $\widehat{\beta}$ be the element of B_n obtained by writing β in terms of the standard Artin generators σ_i of B_n, and then reversing the order of these generators. It is a consequence of the symmetry of the horseshoe map and its inverse that $\beta_{\widehat{P}} = \widehat{\beta'_P}$, where β'_P is the braid obtained by looking at the natural suspension of P from the right rather than from the front of the horseshoe (and is thus conjugate to β_P). The following lemma is an immediate consequence.

Lemma 10. *Let P and R be horseshoe periodic orbits. Then β_P and β_R are conjugate if and only if $\beta_{\widehat{P}}$ and $\beta_{\widehat{R}}$ are conjugate.*

Proof of Theorem 6. Let $w \in \{0,1\}^k$ be a decoration, and let $q \in (0, q_w) \cap \mathbb{Q}$. It is required to prove that the braids β_0 and β_1 with braid codes $c(\beta_0) = c_q 0 w$ and $c(\beta_1) = c_q 1 w$ are conjugate.

By Lemmas 9 and 5, all of the words $c_q{}^0_1\widehat{w}^0_1$ are maximal. Let P and R be the horseshoe periodic orbits with codes $c_P = c_q 0 \widehat{w} 0$ and $c_R = c_q 0 \widehat{w} 1$. Then $\beta_P = \beta_R$ (since $c(\beta_P) = c(\beta_R) = c_q 0 \widehat{w}$). By Lemma 10, it follows that $\beta_{\widehat{P}}$ and $\beta_{\widehat{R}}$ are conjugate. Now $\widehat{P}$ is the periodic orbit containing the point of itinerary $\overline{0w0\widehat{c_q}} = \overline{0w0c_q}$ (since the word c_q is palindromic). Applying Lemma 5 again, $c_{\widehat{P}} = c_q 0 w 0$, and hence $c(\beta_{\widehat{P}}) = c_q 0 w$. Thus $\beta_0 = \beta_{\widehat{P}}$. By the same argument, $\beta_1 = \beta_{\widehat{R}}$, and hence β_0 and β_1 are conjugate as required. □

This proof fails for limit braids, as shown by the following example:

Example 7. Let $w = 110$ and take $q = q_w = 1/3$, so $c_q = 1001$. In the notation of the proof, $c(\beta_0) = 10010110$ and $c(\beta_1) = 10011110$. In fact β_0 and β_1 are conjugate (see Example 8), but this cannot be shown using the method of the proof, since neither of the words $c_q0\widehat{w}_1^0 = 10010011_1^0$ is maximal.

Holmes [Hol89] used similar techniques to study the types of the knots obtained by closing horseshoe braids: two conjugate braids close to the same knot, but two braids which close to the same knot need not be conjugate. Restricted to conjugacy rather than knot equivalence and rewritten in the language of this paper, part of Theorem 3 of [Hol89] becomes:

Theorem 11 (Holmes)**.** *Let v be a palindromic element of $\{0,1\}^m$, and $w \in \{0,1\}^k$ be a word with the property that $1v1_1^0w$ and $1v1_1^0\widehat{w}$ are all braid codes. Then the braids with braid codes $1v1_1^0w$ are conjugate.*

Noting that $c_q = 1v1$ for a palindromic word v, and that if $q < q_w$ then $c_q{}_1^0w$ and $c_q{}_1^0\widehat{w}$ are all braid codes (Lemmas 5 and 9), Holmes's result translates exactly into Theorem 6 (indeed, Table 1 of [Hol89] is precisely a table of decorations). What is added by the treatment here is a description of precisely which conjugacies his theorem gives: if $\beta \in \mathcal{H}$ is a non-limit braid of height q with braid code c, then the only way in which c can be written as $1v1_1^0w$, where v is palindromic, and $1v1_1^0w$ and $1v1_1^0\widehat{w}$ are all braid codes, is by taking $1v1 = c_q$ (this again follows from Lemmas 5 and 9).

If one starts with a limit braid of decoration w and height $q = q_w$, the fact that $c_q{}_1^0\widehat{w}$ need not be braid codes means that larger collections of conjugate horseshoe braids can sometimes be generated by the method of repeated reversal. Since changing the last symbol of c_P doesn't change β_P, it follows from Lemma 10 that any periodic orbit Q whose code can be obtained by reversing that of P an even number of times and changing the final symbol after any subset of these reversals has a representative braid β_Q which is conjugate to β_P.

Example 8. Take $c_P = 100101100$. Then $\beta_P = \beta_Q$, where $c_Q = 100101101$. Reversing both codes gives $c_{\widehat{P}} = 100110100$ and $c_{\widehat{Q}} = 100110110$. The braids $\beta_{\widehat{P}}$ and $\beta_{\widehat{Q}}$ are therefore conjugate. Moreover, $\beta_{\widehat{P}} = \beta_R$ where $c_R = 100110101$, and $\beta_{\widehat{Q}} = \beta_S$, where $c_S = 100110111$. Reversing these codes gives $c_{\widehat{R}} = 100110101 = c_R$, and $c_{\widehat{S}} = 100111101$. Hence $\beta_{\widehat{R}}$ and $\beta_{\widehat{S}}$ are both conjugate to β_P: moreover, since $\beta_{\widehat{R}} = \beta_R$, these braids are also conjugate to $\beta_{\widehat{Q}}$. No further conjugacies can be generated, since if the final symbol of $c_{\widehat{S}}$ is changed to give $c_T = 100111100$, then $c_{\widehat{T}} = c_T$.

In summary, this argument shows that the limit braids with braid codes 10010110, 10011010, 10011011, and 10011110 are all conjugate. Since these correspond to the three decorations 110, 010, and 011, it would follow if Conjecture 8 holds that the six braids with braid codes $c_q{}_1^0110$, $c_q{}_1^0010$, and $c_q{}_1^0011$ are all conjugate for each value of $q \in (0, 1/3) \cap \mathbb{Q}$.

Lemmas 5 and 9 (together with separate simple arguments for NBT and finite order braids) show that this method can only give rise to conjugacies between braids with distinct decorations in the case of limit braids.

References

[Boy84] P. Boyland. Braid types and a topological method of proving positive entropy. Preprint, Boston University (the results of this preprint are included in the published paper [Boy94]), 1984.

[Boy94] P. Boyland. Topological methods in surface dynamics. *Topology Appl.*, 58(3):223–298, 1994.

[dCH] A. de Carvalho and T. Hall. The forcing relation for horseshoe braid types. To appear in *Experiment. Math.*

[Dev89] R. Devaney. *An introduction to chaotic dynamical systems.* Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, second edition, 1989.

[Hal] T. Hall. Software available from `http://www.liv.ac.uk/~tobyhall/hs/`.

[Hal94] T. Hall. The creation of horseshoes. *Nonlinearity*, 7(3):861–924, 1994.

[Hol89] P. Holmes. Knotted periodic orbits in suspensions of Smale's horseshoe: extended families and bifurcation sequences. *Phys. D*, 40(1):42–64, 1989.

[HW85] P. Holmes and R. Williams. Knotted periodic orbits in suspensions of Smale's horseshoe: torus knots and bifurcation sequences. *Arch. Rational Mech. Anal.*, 90(2):115–194, 1985.

[Sma67] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

Institute for Mathematical Sciences, SUNY, Stony Brook, NY 11794-3660, USA

E-mail address: `andre@math.sunysb.edu`

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL, UK

E-mail address: `t.hall@liv.ac.uk`