MAT 530 SPRING 17 HOMEWORK 1

Due Wednesday, Sept 6

Problems in Munkres Text:
Section 13: 4, 8(a)
Section 16: 8

1. Two metrics d_1 and d_2 on a set X are called *equivalent* if there are constants $c, C > 0$ such that

$$cd_1(x,y) \leq d_2(x,y) \leq Cd_1(x,y),$$

for all $x, y \in X$. Prove that equivalent metrics induce the same topology on X.

2. Let $C = C^0([0,1])$ be the space of continuous real-valued functions on $[0,1]$. For $f, g \in C$, define

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx, \quad d_{sup}(f,g) = sup_{x \in [0,1]}|f(x) - g(x)|.$$

Show that d_1 and d_{sup} are metrics on C.

Prove that the topologies on C induced by these metrics are *different*. Is one of them finer than the other?

3. An arithmetic progression is a set of form

$$S(a,b) = \{an + b : n \in \mathbb{Z}\},$$

with $a, b \in \mathbb{Z}$. Define a subset $U \subset \mathbb{Z}$ to be open if it is either empty, or a union of arithmetic progressions.

(a). Show that this defines a topology on $\mathbb{Z}$ in which every non-empty open set is infinite.

(b) Prove the identity

$$\mathbb{Z} \setminus \{-1, 1\} = \cup_{p\ prime} S(p, 0).$$

Prove that each $S(p,0)$ is a closed set while $\mathbb{Z} \setminus \{-1,1\}$ is not closed. Show this leads to a contradiction if there are only finitely many prime numbers. (Hillel Furstenberg proof of Euclid's theorem).