FINAL EXAM, MAT 362 SPRING 14

This is an open book exam, based on the honor system. You can use any books, lecture notes, etc. to assist you in solving the problems. However, you cannot talk or discuss any issues related to the exam with someone else. The exam must express your own, and only your own, understanding.

Total Points: 140.

The exam is due:

Tuesday, May 13, 12 Noon.

Please bring your exam to my office - Math Tower 4-110 at that time. Alternately, you may put the exam under my office door any time prior to the due time.

If you have any questions, you can e-mail me at: anderson@math.sunysb.edu or call at 358-8194.

1. (10pts) Let (ϕ, θ) be spherical coordinates for the unit sphere $S^2(1)$ in $\mathbb{R}^3$. So one has the usual spherical coordinate or latitude/longitude chart:

$$\sigma(\phi, \theta) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta).$$

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$, given by

$$f(x, y, z) = (x - y)^2 + z^2,$$

restricted to the surface $S^2(1)$.

(a). Find the local form of the function f in the spherical coordinate chart.

(b). Express the gradient of $f : S^2(1) \to \mathbb{R}$ in terms of the chart basis vectors.

2. (15 pts) Recall that the coefficients of the first and second fundamental forms of a surface S in a given chart are called E, F, G and L, M, N respectively.

Prove that there is no surface S in $\mathbb{R}^3$ which has a chart for which these coefficients have the form:

$$E = G = 1,\ F = 0 \ \text{ and } \ L = 1,\ M = 0,\ N = -1.$$

3. (20 pts) Find a function $f(u, v)$ of the form $f(u, v) = au + bv + cu^2 + duv + ev^2$ for which the surface given as the graph of f has

$$E = 5/4,\ F = 1/2,\ G = 2,\ L = 3/4,\ M = -3/2,\ N = 3$$

at $(u, v) = (0, 0)$. Here the chart for S is $\mathbf{x}(u, v) = (u, v, f(u, v))$.

Determine also the principal curvatures, corresponding principal vectors, and find the Gauss and mean curvatures K and H of the surface at $(u, v) = (0, 0)$.

4. (15 pts) Let S_1 and S_2 be two surfaces in $\mathbb{R}^3$, with points $p_i \in S_i$. Let $\mathbf{x}_i$ be geodesic polar coordinate charts about p_i on S_i, with coordinates (r, θ), $r < r_0$, $\theta \in [0, 2\pi]$. Prove that if

$$K_{S_1}(r, \theta) = K_{S_2}(r, \theta),$$

for $r < r_0$ and all θ, then S_1 is isometric to S_2 in the common domain $r < r_0$ of the charts.

5. (25 pts) The Euclidean metric on $S = \mathbb{R}^2 \setminus \{0\}$ in (geodesic) polar coordinates is given by

$$g_{Eucl} = dr^2 + r^2 d\theta^2.$$

For $\beta \in (0,1)$, consider the metric

$$g_\beta = r^{-2\beta}(dr^2 + r^2 d\theta^2)$$

conformal to g_{Eucl} on S.

(a). Compute the Gauss curvature of g_β.

(b). Find the routes of the geodesics through the origin $\{0\}$ for the metric g_β in these coordinates.

(c). Show that this surface S with metric g_β is isometric to a surface in $\mathbb{R}^3$; in fact draw a picture of this surface in $\mathbb{R}^3$ for two values (of your choice) of $\beta \in (0,1)$.

(d). Describe what happens to the surface as $\beta \to 1$. Is the limit $\beta = 1$ a surface in $\mathbb{R}^3$?

6. (20pts) Find the total Gauss curvature

$$\int_S K dA,$$

where S is the surface in $\mathbb{R}^3$ given as the set of points (x, y, z) satisfying the equation

$$x^4 + y^4 + z^4 = 16.$$

7. (35 pts) The surface obtained by rotating the curve

$$y = \cosh x,$$

about the x-axis is called the catenoid.

(a). Find the Gauss curvature of the catenoid.

(b). Now find the area element dA and compute, from (a), the total Gauss curvature

$$\int K dA.$$

(You may use the fact that the antiderivative (integral) of $1/\cosh^2$ is tanh.

(c). Describe the image of the Gauss map of the catenoid on the sphere $S^2(1)$, and explain how you can recover your result from (b) almost directly, without computation.

(d). Prove (or at least show why) the catenoid is diffeomorphic to the cylinder, obtained by rotating the line $y = 1$ about the x-axis.

(e). Determine, by any method, the total Gauss curvature, i.e.

$$\int K dA$$

for the cylinder.

(f). Show that the Euler characteristic of the cylinder is 0. You can use the fact that a finite interval $I = (0,1)$ is diffeomorphic to the whole line $\mathbb{R}$, so the cylinder is diffeomorphic to $I \times S^1$.

Conclude from your work above that the Gauss-Bonnet theorem does not hold for non-compact (geodesically complete) surfaces in the same way that it holds for compact surfaces.