FINAL EXAM, MAT 362 SPRING 09

This is an open book exam, based on the honor system. You can use any books, lecture notes, etc. to assist you in solving the problems. However, you cannot talk to or discuss any issues related to the exam with someone else. The exam should reflect completely your own understanding.

The exam is due:

Tuesday, May 19, 1:30pm.

Please bring your exam to my office, Math Tower 4-110, at that time or any time/day before then. If I am not in the office, slide your exam under my office door. You can also contact me to set up a meeting time to hand in your exam if you like.

If you have any questions, e-mail me at: anderson(at)math.sunysb.edu or call at 689-3406.

1. (20pts) Consider the set of points S in $\mathbb{R}^3$ given by

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

(a). Prove that S is a smooth surface.

(b). Show that

$$\mathbf{x}(u,v) = (au\cos v, bu\sin v, u^2),\ \ u > 0$$

is a local chart for S.

(c). Set $a = 1$, $b = 2$ and sketch the surface. Sketch on the surface some lines where $u = const$ and $v = const$, i.e. the local coordinate curves on S induced by $\mathbf{x}$.

2. (20pts) Suppose $\mathbf{x}$ is a conformal chart for a surface, so that the first fundamental form in this chart has coefficients

$$E = G = \lambda(u,v),\ \ F = 0,$$

where $\lambda(u,v)$ is a function of u, v. Show that the Gauss curvature is given by

$$K = -\frac{1}{2\lambda}\Delta(\log\lambda),$$

where Δ is the Laplacian on $\mathbb{R}^2$: $\Delta\phi = (\partial^2\phi/\partial u^2) + (\partial^2\phi/\partial v^2)$.

3. (20pts) Show that there are no compact, (i.e. closed and bounded), minimal surfaces in $\mathbb{R}^3$. Recall a surface is minimal if its mean curvature is 0 everywhere, ($H = 0$).

4. (20pts) Suppose $F : S_1 \to S_2$ is a local isometry from the surface S_1 to S_2. Prove that F maps any geodesic in S_1 to a geodesic in S_2, i.e. if γ is a geodesic in S_1, then $F \circ \gamma$ is a geodesic in S_2.

5. (20pts) Find the total Gauss curvature

$$\int_S K dA,$$

where S is the surface in $\mathbb{R}^3$ given by the equation

$$x^4 + y^6 + z^2 = 1.$$

6. (40pts) The surface obtained by rotating the curve

$$y = \cosh x,$$

about the x-axis is called the catenoid.

(a). Compute the Gauss curvature of the catenoid.

(b). Now find the area element dA and compute, from (a), the total Gauss curvature

$$\int K dA.$$

(You may use the fact that the antiderivative (integral) of $1/\cosh^2$ is tanh.

(c). Describe the image of the Gauss map of the catenoid on the sphere $S^2(1)$, and explain how you can recover your result from (b) almost directly, without computation.

(d). Prove (or at least show why) the catenoid is diffeomorphic to the cylinder, obtained by rotating the line $y = 1$ about the x-axis.

(e). Determine, by any method, the total Gauss curvature, i.e.

$$\int K dA$$

for the cylinder.

(f). Show that the Euler characteristic of the cylinder is 0. You can use the fact that a finite interval I is diffeomorphic to the whole line $\mathbb{R}$, so the cylinder is diffeomorphic to $I \times S^1$.

After all this, you've shown, by examples, that the Gauss-Bonnet theorem does not hold for non-compact surfaces in the same way that it holds for compact surfaces.