Homework 9 Solutions

5.1
(2)

In Polar Coordinate the wave equation looks like

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where $\boxed{x = r\cos\theta,\ y = r\sin\theta}$

So we have the additional conditions

$$u(r, \theta+2\pi, t) = u(r, \theta, t)$$

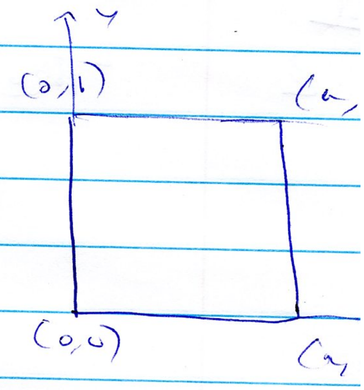
Stays same $\rightarrow$ $\left\{ \begin{array}{l} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{array} \right\}$, $\{ u(a,\theta,t) = 0 \}$

as u vanish on the boundary.

5.2

5) The eqn looks like

$$\nabla^2 u = \frac{1}{k} \frac{\partial u}{\partial t}$$



$$\left(\begin{array}{c} 0 < x < a \\ 0 < y < b \\ 0 < t \end{array} \right)$$

We have the following boundary / initial conditions

From eqns

$$\frac{\partial u}{\partial x}(0,y,t) = 0 \quad (0 \le y \le b), \qquad \frac{\partial u}{\partial y}(x,b,t) = 0 \quad (0 < x < a)$$

$$u(x,0,t) = T_0 \quad (0 < x < a) \quad \text{and} \quad u(a,y,t) = T_0$$

For $t > 0$

and $u(x,y,0) = f(x,y)$ $\quad 0 < x < a,$ $\quad 0 < y < b$

5.3

5) For the new boundary conditi
the EVP for X is

$$X'' + \mu^2 X = 0, \quad X'(0) = 0$$

and

$$X'(a) = 0$$

$\lambda_0 = 0$, $X_0(x) = 1$

We have $X_n(t) = \cos(\mu_n x)$, $(\mu_n = $

The solution is given by equation (
(except for the fact that index n start fr

$$u(x,y,t) = \sum_{n=1}^{\infty} a_{0n} \sin(\nu_n y) e^{-\nu_n^2 kt}$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \cos(\mu_m x) \sin(\nu_n y) e^{-\lambda^2_{mn} t}$$

(where $\lambda^2_{0n} = \nu_n^2$)

6)

The steady state solution

$$V(x,y) = \lim_{t\to\infty} u(x,y,t)$$

V satisfies

$$\nabla^2 v = 0, \quad V(x,0) = f_1(x), \quad v(x,b) = f_2(x)$$
$$V(0,y) = g_1(y), \quad V(a,y) = g_2(y)$$

We can break up V in the following way

$$V(x,y) = V_1(x,y) + V_2(x,y)$$

Then V_1 satisfies

$$\nabla^2 V_1 = 0, \quad V_1(x,0) = 0, \quad V_1(x,b) = 0, \quad V_1(0,y) = g_1(y), \quad V_1(a, \ldots = g_{\ldots}$$

and V_2 satisfies

$$\nabla^2 V_2 = 0, \quad V_2(x,0) = f_1(x), \quad V_2(x,b) = f_2(x)$$
$$V_2(0,y) = 0, \quad V_2(a,y) = 0$$

We can solve V_1, V_2 by methods of 4.2 now.

potential equation

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$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} \qquad 0<x<a, \quad 0<y<b$$

$$\frac{\partial u}{\partial x}(0,y,t)=0, \quad \frac{\partial u}{\partial x}(a,y,t)=0$$

$$0<y<b, \quad 0<t$$

$$\frac{\partial u}{\partial y}(x,0,t)=0, \quad \frac{\partial u}{\partial y}(x,b,t)=0$$

$$0<x<a, \quad 0<t$$

$$u(x,y,0) = f(x,y) = xy$$

u is given by $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\lambda_{mn}^2 t}$

so

$$\therefore \quad a_{00} = \frac{1}{ab}\int_0^b\int_0^a xy\,dx\,dy = \frac{1}{ab}\,\frac{a^2}{2}\cdot\frac{b^2}{2} = \frac{ab}{4}$$

Integrals are computed by Fubini's Theorem

and (Just use Fubini Theorem)

$$a_{m0} = \frac{2}{ab}\int_0^b\int_0^a xy\cos\left(\frac{m\pi x}{a}\right)dx\,dy$$

$$= \frac{-ab\,(1-\cos(m\pi))}{m^2\pi^2}$$

Similarly

$$a_{0n} = \frac{2}{ab}\int_0^b\int_0^a xy\cos\left(\frac{n\pi y}{b}\right)dx\,dy = \frac{-ab\,(1-\cos(n

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The nodal lines are given by

$u_{mn}(x,y,t)=0$

i.e. $\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)=0$

So the lines are given by

$x=0, \frac{a}{m}, \frac{2a}{m}, \ldots$

and

$y=0, \frac{b}{n}, \frac{2b}{n}, \ldots$