MAT-341 HW8 SOLUTIONS

4.4

27) Ex. 21 says

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x') \frac{y}{y^2+(x-x')^2}\,dx'$$

Here

$$u(x,0) = f(x) = \begin{cases} 1 & 0<x \\ 0 & x<0 \end{cases}$$

So $u(x,y) = \frac{1}{\pi}\int_0^{\infty} \frac{y}{y^2+(x-x')^2}\,dx'$

$$(\ast\ast) \quad = \frac{1}{\pi}\int_{-x}^{\infty} \frac{y}{y^2+x_0^2}\,dx_0 \qquad (x_0 = x'-x$$

$$= \frac{1}{\pi} \left| \tan^{-1}\left(\frac{x_0}{y}\right)\right|_{-x}^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right)\right)$$

$(\ast\ast)$ We know $\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

So $\int \frac{a}{x^2+a^2} = \tan^{-1}\left(\frac{x}{a}\right)$

28) Like 27) Here

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\,\frac{y}{y^2+(x-x')^2}\,dx'$$

$$= \frac{1}{\pi}\int_{-a}^{a} \frac{y}{y^2+(x-x')^2}\,dx'$$

$$= \frac{1}{\pi}\int_{-a-x}^{a-x} \frac{y}{y^2+(x_0)^2}\,dx_0 \qquad x_0 = x'-x$$

$$= \frac{1}{\pi}\left|\tan^{-1}\left(\frac{x_0}{y}\right)\right|_{-(a+x)}^{(a-x)}$$

$$= \frac{1}{\pi}\left(\tan^{-1}\left(\frac{a-x}{y}\right) - \tan^{-1}\left(-\frac{(a+x)}{y}\right)\right)$$

7.5 2)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}=0 \quad 0<r<c$$

$$V(0,\theta)=\theta \qquad -\pi<\theta<\pi$$

$$V(r,\theta+2\pi)=V(r,\theta) \qquad 0<r<c$$

$V(r,\theta)$ bounded as $r\to 0^+$

Let $V(r,\theta)=R(r)Q(\theta)$

$$\rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(Q)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2(R(r)Q(\theta))}{\partial\theta^2}=0$$

$$\hookrightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{\phi''(\theta)}{Q(\theta)}=0$$

So

$$\Rightarrow \frac{\phi''}{Q}=-\lambda^2 \qquad \text{as} \quad V \text{ is bounded}$$

Choose $\lambda=n$

$\therefore Q_n(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta)$

and $R_n(r)=r^{\lambda n}=r^n$

So $V(r,\theta)=a_0+\sum(A_n\cos(n\theta)+B_n\sin(n\theta))r^n$

The Boundary Condition Says

$$V(c,\theta)=a_0+\sum(A_n\cos n\theta+B_n\sin(n\theta))c^n=\theta$$

$$-\pi<\theta<\theta$$

So $B_n c^n = \frac{2}{\pi}\int_0^\theta \theta\sin(n\theta)\,d\theta$

$= \frac{2}{n\pi}\cos(n\pi)$ (This integra has become routine now)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta \, d\theta = 0$$

odd

$$a_n = \frac{1}{\pi}\int_{-\theta}^{\theta} \theta \cos(n\theta)\, d\theta = 0$$

$$\therefore \; V(r,\theta) = \sum_{n=1}^{\infty} \frac{-2\cos(n\pi)}{n c^n} \sin(n\theta)\, r^n \longrightarrow (S)$$

The Boundary Condition is not Satisfied at $\theta = \pm\pi$

4. Following 2

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0$$

$$c^n b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta$$

$$= \frac{1}{\pi}\left\{ \int_0^{\pi} \sin(n\theta)\, d\theta - \int_{-\pi}^{0} \sin n\theta \, d\theta \right\}$$

sin is o

$$= \frac{2}{\pi}\int_0^{\pi} \sin(n\theta)\, d\theta$$

$$= \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

$$\therefore \; b_n = \frac{2}{n c^n \pi}\left(1 - (-1)^n\right)$$

$$c^n a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$= \frac{1}{\pi}\left\{ \int_0^{\pi} \cos(n\theta)\, d\theta - \int_{-\pi}^{0} \cos(n\theta)\, d \right\}$$

cos is even

$$= 0$$

Like (S)

$$V(r,\theta) = \sum_{n=1}^{\infty} \frac{2}{n c^n \pi}\left(1 - (-1)^n\right) \sin(n\theta)\, r^n$$

$$V(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^{-n}\left(a_n \cos(n\theta) + b_n \sin(n\theta\right.$$

9

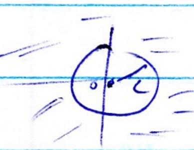
$$\frac{dV}{dr} = \sum_{n=1}^{\infty} -n r^{-(n+1)}\left(a_n \cos(n\theta) + b_n \sin(\right.$$

$$r\frac{dV}{dr} = \sum_{n=1}^{\infty} -n r^{-n}\left(a_n\cos(n\theta) + b_n \sin\right.$$

$$\therefore \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \sum_{n=1}^{\infty} +n^2 r^{-(n+1)} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

$$\frac{dV}{d\theta} = \sum_{n=1}^{\infty} n r^{-n}\left(-a_n \sin(n\theta) + b_n \cos(n\right.$$

$$\frac{\partial^2 V}{\partial\theta^2} = \sum_{n=1}^{\infty} n^2 r^{-n}\left(-a_n \cos(n\theta) - b_n \sin\right.$$



So $\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{\partial^2 V}{\partial \theta^2} = 0$

for $r > c$

$\therefore$ $V(r,\theta)$ is a solution of laplace equati

$$|V(r,\theta)| \le |a_0| + \sum_{n=1}^{\infty} |r^{-n}|\,|a_n\cos(n\theta) + b_n S$$

as $r \to \infty$ $\quad |r^{-n}|\,|a_n \cos(n\theta) + b_n \sin(n\theta)| \to 0$

So $\lim_{r\to\infty} |V(r,\theta)|$ is bounded

10

It is Clear from exercise 9

$V(c,\theta)$

$$a_0 + \sum_{n=1}^{\infty} \frac{1}{c^n}\left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$

$$= f(\theta)$$

So $\frac{a_n}{c^n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$ for $n \geq 1$

$$\Rightarrow \quad a_n = \frac{c^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

Also $\frac{b_n}{c^n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$

So $b_n = \frac{c^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$