Homework 10 Solutions

5.9

(11) $f(x) = |x|$, f is even

$\therefore$ $b_n = 0$ for n odd

$b_n = (2n+1) \int_0^1 x P_n(x)\, dx$

(I.B.P) $= \dfrac{(2n+1)}{(n+2)(n-1)} \left[(1-x^2)\left(P_n(x) - x P_n'(x)\right) \right]_0^1$

$= \dfrac{2n+1}{(n+2)(n-1)} \left(-P_n(0)\right)$

$= \dfrac{2n+1}{(n+2)(n-1)} (-1)^{n/2} \dfrac{1 \cdot 3 \cdots (n-1)}{2 \cdots n}$

for $n = 2, 4, 6 \ldots$

$\therefore$ $b_n = \frac{1}{2}$ for $n = 0$

$\therefore$ we have $f(x) = \sum_{n=0}^{\infty} b_n P_n(x)$

(12)

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

$$f - \frac{1}{2} = \begin{cases} -\frac{1}{2} & -1 < x < 0 \\ \frac{1}{2} & 0 < x < 1 \end{cases}$$

$f - \frac{1}{2}$ is odd

so $b_n = 2, 4, \ldots$

$$b_n = (2n+1) \int_0^1 \frac{1}{2} P_n(x)\, dx$$

$$= \frac{(2n+1)}{2n(n+1)} \left[(1-x^2) P_n'(x) \right]_0^1$$

$$= \frac{(2n+1)}{2(n+1)} (-1)^{\frac{n-1}{2}} \left(\frac{1 \times 3 \cdots (n-2)}{2 \times 4 \times \cdots \times (n-1)} \right)$$

for $n = 3, 5, 7, \ldots$

$$b_1 = 3 \int_0^1 \frac{1}{2} x \, dx = \frac{3}{2}$$

So $f(x) = \frac{1}{2} + \frac{3}{2} P_1(x) + \cdots$

5.10

(1) $u(1,\phi) = \begin{cases} 1 & 0<\phi<\frac{\pi}{2} \\ 0 & \frac{\pi}{2}<\phi<\pi \end{cases}$

$$u(\rho,\phi) = \sum_{n=0}^{\infty} b_n \rho^n P_n(\cos(\phi))$$

$$b_n = \frac{2n+1}{2}\int_0^{\pi/2} P_n(\cos\phi)\sin\phi \, d\phi$$

$$= \frac{2n+1}{2}\int_0^1 P_n(x)\, dx \qquad (x=\cos\phi)$$

$$= \frac{2n+1}{2n(n+1)}\left[-(1-x^2)P_n'(x)\right]_0^1$$

So $b_0 = \frac{1}{2}$

$b_n = 0$ for $n = 2, 4, \dots$

$b_n = -\frac{2n+1}{2(n+1)}(-1)^{n/2}\frac{1\cdot 3\cdots(n-2)}{2\cdot 4\cdots(n-1)}$

for $n = 3, 5, \dots$

and again

$b_1 = \frac{3}{2}\int_0^1 x \, dx = \frac{3}{4}$

u is given by

$$u(\rho,\phi) = \sum_{n=0}^{\infty} b_n \rho^n P_n(\cos(\phi))$$

(2) $\boxed{\nabla^2 u = 0}$

Like the last problem we write the general so'n

$$u(\rho, \phi) = \sum_{n=0}^{\infty} b_n \rho^n P_n(\cos\phi)$$

$$b_n = (2n+1) \int_0^{\pi/2} u(1,\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

$x = \cos\phi$

$$= (2n+1) \int_0^1 P_n(x) \, dx$$

See last problem

for $n = 3, 5, \ldots$

$$= (2n+1)(-1)^{n-1/2} \frac{1 \cdot 3 \cdots (n-2)}{2 \cdot 4 \cdots (n-1)}$$

$b_1 = \frac{3}{2}$

$b_n = 0$ for $n = 0, 2, 4, \ldots$

(19)

$$\phi(x,y) = \sin(\pi x)\sin(2\pi y) - \sin(2\pi x)\sin(\pi y)$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

$$= -\pi^2 \sin(\pi x)\sin 2\pi y + 4\pi^2 \sin(2\pi x)$$

$$- 4\pi^2 \sin(\pi x)\sin 2\pi y + \pi^2 \sin 2\pi x \, s$$

$$= -5\pi^2\phi$$

at $y=0$, $\phi = 0$

At $y=1$, $\phi = \sin(\pi x)\sin 2\pi x - \sin 2\pi$

at $x=1$ $\phi = 0$ ($\Rightarrow \sin\pi$

(25) $\nabla^2 u = \frac{\partial u}{\partial t}$

$\left(u(r,t) = t^b e^{-r^2/4t}\right) \Rightarrow \frac{\partial u}{\partial t} = \frac{b}{t} u$

in polar coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{(-2r)}{4t} t^2 e^{-r^2/4t}\right)$$

$$= -\frac{1}{2rt}\frac{\partial}{\partial r}(r^2 u) = -\frac{1}{2rt}\left(2ru + r^2\frac{\partial u}{\partial r}\right)$$

$$= -\frac{1}{2rt}\left(2ru + r^2 \cdot \frac{-2r}{4t} u\right)$$

$$= u\left(\frac{-1}{t} + r^2/4t^2\right) \text{ —— (2)}$$

Comparing (1) and (2) we get $b=-1$