PRACTICE FINAL EXAM

MAT 341, Fall 13

1. (a) Find the Fourier series of the function $f(x)$, periodic of period 2π, equal to 0 on $[0, \pi/2]$, 1 on $[\pi/2, 3\pi/2]$ and 0 on $[3\pi/2, 2\pi]$.

(b) Use this Fourier series to deduce the following series representation for π:

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots).$$

2. Find the solution of the Laplace equation

$$\Delta u = 0,$$

in the unit disc $D_1 \subset \mathbb{R}^2$, with boundary function

$$u(1, \theta) = \cos\theta \cdot \sin\theta.$$

3. The disc $D_1 \subset \mathbb{R}^2$ of radius 1 is completely insulated (0 temperature flux through the boundary). The initial temperature distribution is

$$u(r, \theta, 0) = \cos \pi r.$$

What is the steady-state temperature distribution?

4. Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

for $0 < x < \pi$ with boundary and initial conditions:

$$u(0, t) = 0, \quad u(\pi, t) = 0,$$

$$u(x, 0) = 2\sin 3x, \quad \frac{\partial u}{\partial t}(x, 0) = 0.$$

5. The motion of a string of length π, fixed at both ends and vibrating in a viscous medium is governed by the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t}$$

(the damped wave equation) with boundary conditions

$$u(0, t) = 0, \quad u(\pi, t) = 0.$$

Here k measures the viscosity of the medium ($k = 0$ is the standard wave equation) and we have set $c = 1$.

Set $k = 4$ and suppose the initial conditions for the displacement of the string are

$$u(x, 0) = \sin x, \quad \partial_t u(x, 0) = 0.$$

(a). Set $u(x, t) = \phi(x)T(t)$ as usual to separate the variables and find the associated equations for T and ϕ.

(b). Solve the equation for ϕ with given initial condition and then solve the resulting equation for T (You need to solve the damped harmonic oscillator equation as in p.5 of the text.

(c). Now find the full solution $u(x,t)$.

(d). What happens to the string as $t \to \infty$?