MATH 308 Final Exam

May 23, 2011

Name: ______________________ **ID:** __________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	0	20	20	20	20	20	20	20	20	160
Score:										

There are 9 problems on 9 pages in this exam (not counting the cover sheet). Make sure that you have them all.

You **may use a calculator** if you wish, provided your calculator does not do calculus. However, it is unlikely to be of much help.

You are **allowed to use a single page of notes**, written on the front an the back, during this exam.

You should turn in your **take-home problem** to the exam. Make sure that you write your name on each page of your solution to the take-home problem.

Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. **Books, extra papers (other than the sheet of notes), and discussions with friends are not permitted.**

You may choose to skip **any one problem** on the exam. Please cross off the problem you don't want graded.

There is a table of Laplace transforms on the next page. Use it, or don't. Applying the inverse Laplace transform to turn annoying algebraic calculations into "simpler" calculus problems probably won't work, but feel free to try. Just don't make me grade your attempts.

You have a bunch of time to do this exam. I forget how much, but it seems really, really long to me, and probably way too short to you.

Table of Laplace Transforms

$f(t)$ for $t \geq 0$	$\mathcal{L}[f](s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ $(n = 0, 1, \ldots)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2-\omega^2)^2}$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$\delta(t-a)$	e^{-as}
$H_a(t)$	$\frac{e^{-as}}{s}$
$H_a(t) f(t-a)$	$e^{-as}\mathcal{L}[f](s)$
$e^{at} f(t)$	$\mathcal{L}[f](s-a)$
$f'(t)$	$s\mathcal{L}[f](s) - f(0)$
$f''(t)$	$s^2\mathcal{L}[f](s) - sf(0) - f'(0)$

Recall that $H_a(t)$ is the Heaviside function, $H_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

and $\delta(t)$ is the Dirac δ-function[1], with $\delta(t) = 0$ for all $t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)dx = 1$.

[1]although it is not really a function, but a measure

1. **Take-home problem.** Give a qualitative analysis of the system

$$\frac{dx}{dt} = b + y$$
$$\frac{dy}{dt} = -\sin(x) - ay$$

where a and b are real constants.

You should find and classify all equilibria, and say something about the behavior of the solutions. Where relevant, say something about eigenvalues and or eigenvectors. Pay particular attention to the bifucations that occur as a and b vary. Note that a and b can be positive, negative, or zero.

Your classification should describe what happens for all values of a and b; something along the lines of the bifurcation diagram for 2×2 systems in terms of trace and determinant is probably relevant.

You are welcome (even encouraged) to use relevant technology such as Maple, Mathematica, Matlab, or the software on the class web page to help you analyze the system. You are encouraged to include relevant pictures in your solution.

Some possibly relevant background:

This system models the behavior a damped pendulum which has a propeller attached to it, which adds a constant angular velocity b (this could be arranged so that b is in the direction of x or away from it, corresponding to $b > 0$ or $b < 0$.

Here, $x(t)$ represents the angle (in radians) that the pendulum makes with the vertical, and $y(t)$ is the angular velocity. It may help you to interpret the solutions in terms of the behaviour of the pendulum.

In class, we discussed the case $b = 0$, that is, the damped pendulum where the constant a corresponds to the amount of friction. The case $a > 0$ has "negative friction", which isn't physically realistic but still should be considered.

Recall that bifurcations occur when $a = \pm 2$ and $a = 0$ (we really only discussed the bifurcation at $a = 0$ in any detail). At $a = 2$, the spiral sink at $(0, 0)$ changes to a sink with eigenvalues $\frac{1}{2}\left(a \pm \sqrt{a^2 - 4}\right)$. In terms of the behavior of the pendulum, this means that when $0 < a < 2$, solutions which limit on $x = 0$ do so by oscillating towards straight down, while for $a > 2$ these solutions either limit directly on the equilibrium, or overshoot exactly once and then limit on the equilibrium.

Something similar happens for $a = -2$.

Please include the pages of your solution to this problem with the exam. If you wish to skip this problem, draw a big X over the entire page.

$\boxed{\text{20 pts.}}$ 2. Let $\mathcal{P}_4$ denote the vector space of polynomials of degree 4 or less, that is,

$$\mathcal{P}_4 = \left\{ ax^4 + bx^3 + cx^2 + dx + e \;\middle|\; a, b, c, d, e \in \mathbb{R} \right\}$$

(a) Is the set $U = \left\{ p(x) \in \mathcal{P}_4 \;\middle|\; p(1) = 0 \right\}$ a subspace of $\mathcal{P}_4$? Justify your answer.

(b) Is the set $V = \left\{ p(x) \in \mathcal{P}_4 \;\middle|\; p'(0) = 1 \right\}$ a subspace of $\mathcal{P}_4$? Justify your answer.

(c) Is the set $W = \left\{ p(x) \in \mathcal{P}_4 \;\middle|\; \int_0^1 p(t)dt = 0 \right\}$ a subspace of $\mathcal{P}_4$? Justify your answer.

(d) Choose one of the sets U, V, or W from the previous parts which is a subspace, and give a basis for the subspace. (No credit if it isn't a subspace).

$\boxed{\text{20 pts.}}$ 3. (a) Find $y(t)$ which satisfies $y''(t) = 4y(t)$, with $y(0) = 4$ and $y'(0) = 4$.

(b) Find the most general form of the function $x(t)$ for which $x''(t) - 4x(t) = 16te^{2t}$.

$\boxed{\text{20 pts.}}$ 4. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation for which $f(0,0,1) = (0,0,1)$, and suppose the kernel of f is the plane spanned by $(1,1,1)$ and $(1,-1,1)$. Write the matrix B which represents the transformation f.

(b) Is there a diagonal matrix Λ so that $U^{-1}B\,U = \Lambda$ for some invertible matrix U? If so, give the matrices U and Λ. If not, explain why not.

$\boxed{\text{20 pts.}}$ 5. (a) Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Calculate e^{At}.

(b) Give the general solution to the system of differential equations

$$\frac{dx}{dt} = 3x + y \qquad \frac{dy}{dt} = 3y + z \qquad \frac{dz}{dt} = 3z$$

$\boxed{\text{20 pts.}}$ 6. A function x satisfies $x''' = x^2\, x'$, with $x(0) = 1$, $x'(0) = 1$, $x''(0) = 1$. Find the 4^{th} Taylor polynomial of $x(t)$ at $t = 0$.

7. Let $x(t)$, $y(t)$ solve the system of equations

$$\frac{dx}{dt} = x - y^2 \qquad \frac{dy}{dt} = x + y - 2$$

5 pts.

(a) If $x(0) = 1.1$, $y(0) = 1$, which of the following statements is true?

A. The point $(x(100, y(100))$ will be quite close to $(1, 1)$.

B. The point $(x(100, y(100))$ will be at least 3 units away from $(1, 1)$.

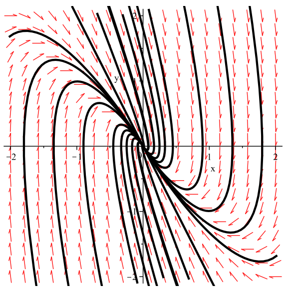
C. Neither of the above is true.

15 pts.

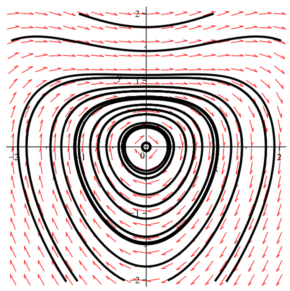
(b) Give a solid justification for your answer above.

$\boxed{\text{20 pts.}}$ 8. Below are five second order differential equations labeled (a) through (e), and four phase portraits labeled 1 through 4 with a number of trajectories drawn. On the line following of each of the equations, write the letter of the corresponding phase portrait or the word "none" if the phase portrait is not shown.

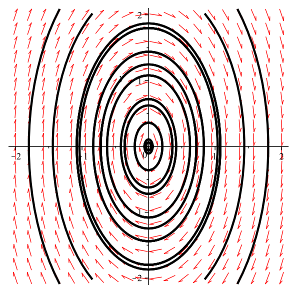
1.



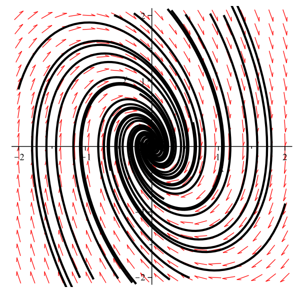
2.



3.



4.



(a) $x'' + 4x' + 4x = 0$

(a) ____________

(b) $x'' + 3x = 0$

(b) ____________

(c) $x'' + x' + 2x = 0$

(c) ____________

(d) $x'' - x' - 6x = 0$

(d) ____________

(e) $x'' - \sin(x)x' + x = 0$

(e) ____________

$\boxed{\text{20 pts.}}$ 9. Suppose $y(0) = 1$, $y'(0) = 1$, and

$$y'' + y = \begin{cases} 1 & \text{if } 0 \le t < 3\pi \\ 0 & \text{if } 3\pi \le t \end{cases}$$

Find $y(t)$.