EXTENSION OF SYMMETRIES ON EINSTEIN MANIFOLDS WITH BOUNDARY

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Abstract. We investigate the isometry extension property for Einstein metrics on manifolds with boundary; namely when Killing fields of the boundary metric extend to Killing fields of any filling Einstein metric.

1. Introduction.

Let M^{n+1} be a compact $(n+1)$-dimensional manifold-with-boundary, and suppose g is a (Riemannian) Einstein metric on M, so that

$$Ric_g = \lambda g, \tag{1.1}$$

for some constant $\lambda \in \mathbb{R}$. The metric g induces a Riemannian boundary metric γ on ∂M. In this paper we consider the issue of whether isometries of the boundary structure $(\partial M, \gamma)$ necessarily extend to isometries of any filling Einstein manifold (M, g).

In general, without any assumptions, this isometry extension property will not hold. It is false for instance if ∂M is not connected. For example, let $M = S^3 \setminus (B_1 \cup B_2)$, where B_i are a pair of disjoint round 3-balls in S^3 endowed with a round metric; then a generic pair of Killing fields X_i on $S^2_i = \partial B_i$ does not extend to a Killing field on M. Also, setting $M = T^3 \setminus B$ where B is a round 3-ball in a flat 3-torus T^3, one sees again that Killing fields on ∂M do not extend to Killing fields on T^3. This is due to the fact that $\pi_1(\partial M)$ does not surject onto $\pi_1(M)$. Both situations above can be remedied by making the topological assumption

$$\pi_1(M, \partial M) = 0, \tag{1.2}$$

so we will usually assume (1.2).

However, this condition is still not sufficient. Consider for example the flat product metric on $S^1 \times \mathbb{R}^2$. Let σ be any simple closed curve in $\mathbb{R}^2$ and let $T_\sigma = S^1 \times \sigma \subset S^1 \times \mathbb{R}^2$. Then T_σ bounds a compact domain $M \subset S^1 \times \mathbb{R}^2$, diffeomorphic to a solid torus. Any such T_σ is flat with respect to the induced metric, and so has a pair of orthogonal Killing fields. One of these, that tangent to the S^1 factor, clearly extends to a Killing field of M, (in fact $S^1 \times \mathbb{R}^2$). However, whenever σ is not a round circle in $\mathbb{R}^2$, (so that σ has non-constant geodesic curvature), the orthogonal Killing field on (T_σ, γ) tangent to σ does not extend as a Killing field to M.

Very similar examples are easily constructed via the Hopf fibration in the sphere $\mathbb{S}^3$, with M again a solid torus in S^3, as first pointed out to the author by H. Rosenberg [13], cf. [5], [8], [12] and references therein for detailed discussion. Similar examples, even with convex boundary, also occur in hyperbolic space-forms, cf. Remark 3.6 below, and in higher dimensions by taking products.

The main result of this paper characterizes one situation where the isometry extension property does hold. Let H denote the mean curvature of ∂M in (M, g).

Theorem 1.1. *Let g be a $C^{m,\alpha}$ Einstein metric on M, $m \geq 3$, with $\lambda \leq 0$, with induced boundary metric γ on ∂M, and suppose (1.2) holds. Then any Killing field X on $(\partial M, \gamma)$ for which $X(H) = 0$, extends uniquely to a Killing field on (M, g).*

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It follows for instance that for $H = const$, the identity component $Isom_0(\partial M, \gamma)$ of the isometry group of $(\partial M, \gamma)$ embeds in the isometry group of any $\lambda \leq 0$ Einstein filling metric (M, g):

$$Isom_0(\partial M, \gamma) \hookrightarrow Isom_0(M, g),$$

or equivalently, such isometries of the boundary extend to isometries of any Einstein filling metric. A simple consequence of Theorem 1.1 is for example the following rigidity result.

Corollary 1.2. *Let g be a $C^{3,\alpha}$ Einstein metric on M^{n+1} which induces the round metric γ_{+1} on the boundary $\partial M = S^n$, $n \geq 2$. If $\pi_1(M) = 0$, $\lambda \leq 0$ and $H = const$, then modulo rescalings, (M, g) is isometric to a standard round ball in a simply connected space form $\mathbb{H}^{n+1}$ or $\mathbb{R}^{n+1}$.*

We expect these results also hold for $\lambda > 0$, cf. Remark 5.1.

There are natural analogs of these results valid for exterior domains. Thus, let M^{n+1} be an open or non-compact manifold with compact "inner" boundary and with a finite number of non-compact ends. Metrically, consider complete metrics g on M which are asymptotically (locally) flat on each end. In this context, Theorem 1.1 also holds for Einstein metrics, cf. Proposition 5.4. A similar result also holds for complete, asymptotically hyperbolic Einstein metrics, with boundary at infinity, without any assumption on the mean curvature, cf. Theorem 5.5.

We point out that Theorem 1.1, (and Corollary 1.2), remain valid without the hypothesis (1.2) provided (M, g) is embedded as a domain in a complete, simply connected Einstein manifold $(\hat{M}, \hat{g})$. It should also be noted that the isometry extension property is false for isometries not contained in $Isom_0(\partial M, \gamma)$. As a simple example, consider a flat metric on a solid torus $M = D^2 \times S^1$ of the form

$$g_0 = dr^2 + r^2 d\theta_1^2 + d\theta_2^2,$$

for $r \in [0, 1]$. Then interchanging the two circles parametrized by θ_1 and θ_2 is an isometry of the boundary, which does not extend to an isometry of the solid torus. Of course ∂M is both convex and has constant mean curvature in (M, g_0).

The proofs of the results above follow from a study of the global properties of the space of Einstein metrics g on M. As shown in [3], the moduli space $\mathcal{E}$ of such metrics is a smooth Banach manifold, for which the (Dirichlet) map to the boundary metrics

$$\Pi_D : \mathcal{E} \rightarrow Met(\partial M), \quad \Pi_D(g) = g_{T(\partial M)}, \tag{1.3}$$

is C^∞ smooth, cf. Theorem 2.1. The main results are then quite simple to prove when the metric (M, g) is non-degenerate, in the strong sense that the derivative $D\Pi_D$ of Π_D at g has trivial kernel, cf. Remark 3.3. They also hold, with somewhat more involved proofs, when $D\Pi_D$ has no cokernel, (or when $ImD\Pi_D$ is dense in $TMet(\partial M)$). In both of the situations above, the results hold without any condition on the mean curvature, i.e. without assuming $X(H) = 0$. The proof of Theorem 1.1 and Corollary 1.2 in general follows from a careful study of the Bianchi-gauged Einstein operator and natural perturbations of this operator.

A brief survey of the contents of the paper is as follows. In §2, we introduce the basic setting and structural results on the space of Einstein metrics, needed for the work to follow. Section 3 studies elliptic boundary value problems and Fredholm properties of the boundary map Π_D in (1.3). Section 4 relates the isometry extension property with the (linearized) constraint equations induced by the Einstein equations on ∂M. In §5, we prove Theorem 1.1 and Corollary 1.2, and the further related results mentioned above.

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2. The Space of Einstein Metrics

As above, let M denote a connected, compact, oriented $(n+1)$-dimensional manifold with compact, non-empty boundary ∂M. Consider the Banach space

$$Met(M) = Met^{m,\alpha}(M) \tag{2.1}$$

of Riemannian metrics on M which are $C^{m,\alpha}$ smooth up to ∂M. Here m is any fixed integer with $m \geq 2$, including $m = \infty$, (giving a Fréchet space), and $\alpha \in (0,1)$. Let

$$\mathbb{E} = \mathbb{E}^{m,\alpha}(M) \subset Met^{m,\alpha}(M) \tag{2.2}$$

be the subset of Einstein metrics on M, $C^{m,\alpha}$ smooth up to ∂M, with

$$Ric_g = \lambda g, \tag{2.3}$$

for λ arbitrary, but fixed, (so that $\mathbb{E} = \mathbb{E}(\lambda)$); Ric_g is the Ricci curvature of g. The smoothness index (m,α) will occasionally be suppressed from the notation when its exact value is unimportant.

The space $\mathbb{E}^{m,\alpha}(M) \subset Met^{m,\alpha}(M)$ is invariant under the action of the group $\mathcal{D}_1 = \mathcal{D}_1^{m+1,\alpha}$ of orientation preserving $C^{m+1,\alpha}$ diffeomorphisms of M equal to the identity on ∂M. This action is free, (since any such isometry equal to the identity on ∂M is necessarily the identity), and well-known to be proper. The moduli space $\mathcal{E} = \mathcal{E}^{m,\alpha}(M)$ of Einstein metrics on M is defined to be the quotient

$$\mathcal{E} = \mathbb{E}/\mathcal{D}_1. \tag{2.4}$$

One has a natural Dirichlet boundary map

$$\Pi_D : \mathbb{E} \to Met(\partial M);\ \Pi_D(g) = \gamma = g|_{T(\partial M)}. \tag{2.5}$$

which clearly descends to a map

$$\Pi_D : \mathcal{E} \to Met(\partial M);\ \ \Pi_D([g]) = \gamma. \tag{2.6}$$

We note the following result, proved in [3].

Theorem 2.1. *Suppose $\pi_1(M, \partial M) = 0$ and $m \geq 3$. Then the space $\mathcal{E}$ is a C^∞ smooth Banach manifold, (Fréchet manifold when $m = \infty$), and the boundary map Π_D is C^∞ smooth.*

Theorem 2.1 is proved by a suitable application of the implicit function theorem. Strictly speaking, this result is not needed for the proof of the main results in the Introduction; however it places the arguments to follow in a natural context.

Consider the Einstein operator

$$\begin{gathered} E : Met(M) \to S_2(M), \\ E(g) = Ric_g - \lambda g, \end{gathered} \tag{2.7}$$

where $S_2(M)$ is the space of symmetric bilinear forms on M. The linearization of E is given by

$$L_E(k) = 2\frac{d}{dt}(Ric_{g+tk} - \lambda(g+tk))|_{t=0} = D^*Dk - 2R(k) - 2\delta^*\beta(k); \tag{2.8}$$

here $\delta^*X = \frac{1}{2}\mathcal{L}_X g$, $\beta(k) = \delta(k) + \frac{1}{2}dtrk$ is the Bianchi operator with respect to g and $R(h)$ is the action of the curvature tensor on symmetric bilinear forms k, cf. [4] for instance.

The tangent space $T_g\mathbb{E}$ is given by $KerL_E$. The derivative of the Dirichlet boundary map Π_D in (2.5) acts on forms k satisfying $L_E(k) = 0$ and is given by

$$(D\Pi_D)_g(k) = k^T|_{\partial M}, \tag{2.9}$$

where k^T is the tangential projection or restriction of k to $T(\partial M)$. Thus k^T is the variation of the boundary metric $\gamma = \Pi_D(g)$. It will also be important to consider the variation of the 2nd fundamental form A of ∂M in M. Thus, analogous to (2.6), one has a natural Neumann boundary map

$$\Pi_N : \mathcal{E} \to S^2(\partial M), \quad \Pi_N([g]) = A. \tag{2.10}$$

This is well-defined, since A is invariant under the action of $\mathcal{D}_1$. Note also that Π_N maps $\mathcal{E}^{m,\alpha}$ to $S_2^{m-1,\alpha}(\partial M)$. To compute the derivative of Π_N, let $g_s = g + sk$ be a variation of g. Since $A = \frac{1}{2}\mathcal{L}_N g$, one has $2A'_k \equiv 2\frac{d}{ds}A_{g_s}|_{s=0} = (\mathcal{L}_N g)' = \mathcal{L}_N k + \mathcal{L}_{N'} g$. A simple computation gives $N' = -k(N)^T - \frac{1}{2}k_{00}N$, where $k(N)^T$ is the component of $k(N)$ tangent to ∂M and $k_{00} = k(N,N)$. Thus

$$A'_k = (D\Pi_N)(k) = \frac{1}{2}(\mathcal{L}_N k + \delta^* V), \tag{2.11}$$

where $V = 2N' = -2k(N)^T - k_{00}N$.

The kernel of $D\Pi_D$ in (2.5) consists of forms k satisfying $L_E(k) = 0$ and $k^T = 0$ on ∂M, while the kernel of $D\Pi_N$ in (2.10) consists of such forms satisfying $(A'_k)^T = 0$ at ∂M. Thus, if both conditions hold,

$$k^T = 0, \ (A'_k)^T = 0 \ \text{ at } \ \partial M, \tag{2.12}$$

then (M, g) is both Dirichlet and Neumann degenerate, i.e. a singular point of each boundary map. We note that each of the conditions in (2.12) is gauge-invariant, i.e. invariant under the addition of terms of the form $\delta^* Z$ with $Z = 0$ on ∂M. Of course any form k satisfying $k = \nabla_N k = 0$ at ∂M satisfies (2.12). Changing such k by arbitrary such gauge transformations shows that (2.12) is equivalent to the statement that k is pure gauge, to first order at ∂M, i.e.

$$k = \delta^* Z + O(t^2), \tag{2.13}$$

near ∂M, with $Z = 0$ on ∂M, where $t(x) = dist_g(x, \partial M)$.

The natural or geometric Cauchy data for the Einstein equations (2.3) on M at ∂M consist of the pair (γ, A). If k is an infinitesimal Einstein deformation of (M, g), so that $L_E(k) = 0$, then the induced variation of the Cauchy data on ∂M is given by k^T and $(A'_k)^T$. It is natural to expect that an Einstein metric g is uniquely determined in a neighborhood of ∂M, up to isometry, by the Cauchy data (γ, A), i.e. one should have a suitable unique continuation property for Einstein metrics. Similarly, one would expect this holds for the linearized Einstein equations. The next result, proved in [2], confirms this expectation.

Theorem 2.2. *Let $g \in \mathbb{E}^{m,\alpha}$, $m \geq 3$, and suppose k is an infinitesimal Einstein deformation which is both Dirichlet and Neumann degenerate, so that $L_E(k) = 0$ and (2.12) holds. Then k is pure gauge near ∂M, i.e.*

$$k = \delta^* Z \ \text{ near } \ \partial M, \tag{2.14}$$

with $Z = 0$ on ∂M.

As is well-known, the operator E is not elliptic, due to its covariance under diffeomorphisms: one has $L_E(\delta^* Y) = 0$, for any vector field Y on M, at an Einstein metric. We will require ellipticity at several points and so need a choice of gauge to break the diffeomorphism invariance of the Einstein equations. In view of (2.8), the simplest and most natural choice for the work to follow is the Bianchi gauge. Thus, let $\widetilde{g}$ be a fixed (background) metric in $\mathbb{E}$. The associated Bianchi-gauged Einstein operator is given by the C^∞ smooth map

$$\Phi_{\widetilde{g}} : Met^{m,\alpha}(M) \to S_2^{m-2,\alpha}(M), \tag{2.15}$$

$$\Phi(g) = \Phi_{\widetilde{g}}(g) = Ric_g - \lambda g + \delta_g^* \beta_{\widetilde{g}}(g),$$

where $\beta_{\widetilde{g}}(g)$ is the Bianchi operator with respect to $\widetilde{g}$, while δ^* is taken with respect to g. Although $\Phi_{\widetilde{g}}$ is defined for all $g \in Met(M)$, we will only consider it acting on g near $\widetilde{g}$.

The linearization of Φ at $\widetilde{g} = g$ is given by

$$L : T_{\widetilde{g}}Met(M) \to S_2(M), \tag{2.16}$$

$$L(h) = 2(D\Phi)_{\widetilde{g}}(h) = D^*Dh - 2R(h).$$

The operator L is formally self-adjoint and is clearly elliptic. Comparing (2.7) and (2.15), the relation between L and the linearization $L_E = 2E'$ of the Einstein operator E in (2.8) is given by

$$L_E = L - 2\delta^*\beta. \tag{2.17}$$

In §3, we will consider elliptic boundary value problems for the operator Φ.

Clearly $g \in \mathbb{E}$ if $\Phi_{\widetilde{g}}(g) = 0$ and $\beta_{\widetilde{g}}(g) = 0$, so that g is in the Bianchi gauge with respect to $\widetilde{g}$. Given $\widetilde{g}$, let $Met_C(M) = Met_C^{m,\alpha}(M)$ be the space of $C^{m,\alpha}$ smooth Riemannian metrics on M in Bianchi gauge with respect to $\widetilde{g}$ at ∂M:

$$Met_C(M) = \{g \in Met(M) : \beta_{\widetilde{g}}(g) = 0 \text{ at } \partial M\}. \tag{2.18}$$

Let

$$Z_C = \{g \in Met_C(M) : \Phi(g) = 0\} \tag{2.19}$$

be the 0-set of Φ and let $\mathbb{E}_C \subset Z_C$ be the subset of Einstein metrics g in Z_C.

To justify the use of Φ, one needs to show that the opposite inclusion holds, so that $\mathbb{E}_C = Z_C$. This has already been done in [3] and we summarize the results here.

Lemma 2.3. **(i).** *For g in $Met^{m,\alpha}(M)$, one has*

$$T_gMet^{m-2,\alpha}(M) \simeq S_2^{m-2,\alpha}(M) = Ker\delta \oplus Im\delta^*, \tag{2.20}$$

where δ^* acts on $\chi_1^{m-1,\alpha}$, the space of $C^{m-1,\alpha}$ vector fields on M which vanish on ∂M.

(ii). *For $\widetilde{g} \in \mathbb{E}^{m,\alpha}$ and g in $Met^{m,\alpha}(M)$ close to $\widetilde{g}$, one has*

$$T_gMet^{m-2,\alpha}(M) \simeq S_2^{m-2,\alpha}(M) = Ker\beta \oplus Im\delta^*, \tag{2.21}$$

where β is the Bianchi operator with respect to $\widetilde{g}$. If $g \in \mathbb{E}^{m,\alpha}$, then (2.21) holds with m in place of $m-2$.

(iii). *Any metric $g \in Z_C$ near $\widetilde{g} \in \mathbb{E}^{m,\alpha}$ is Einstein, and in Bianchi gauge with respect to $\widetilde{g}$, i.e.*

$$\beta_{\widetilde{g}}(g) = 0. \tag{2.22}$$

Similarly, if $k \in Met_C(M)$ is an infinitesimal deformation of $\widetilde{g}$ in Z_C, i.e. $L(k) = 0$, then k is an infinitesimal Einstein deformation and $\beta(k) = 0$.

Lemma 2.3 implies that $\mathbb{E}_C = Z_C$ near $\widetilde{g}$, and at least infinitesimally $\mathbb{E}_C$ is a local slice for the action of the diffeomorphism group $\mathcal{D}_1$ on $\mathbb{E}$. In fact, it is shown in [3] that $\mathbb{E}_C$ is a local slice for the action of $\mathcal{D}_1$.

The next result is a preliminary version of Theorem 1.1.

Proposition 2.4. *Let $g \in \mathbb{E}^{m,\alpha}$, $m \geq 3$, and suppose X is a Killing field on $(\partial M, \gamma)$ such that*

$$(\mathcal{L}_X A)^T = 0 \text{ at } \partial M, \tag{2.23}$$

If $\pi_1(M, \partial M) = 0$, then X extends to a Killing field on (M, g).

Proof: Since $\gamma \in Met^{m,\alpha}(\partial M)$, the Killing field X is $C^{m+1,\alpha}$ smooth on ∂M. By Lemma 2.3, X may be uniquely extended to a vector field X on M so that

$$\beta\delta^*X = 0 \text{ on } M. \tag{2.24}$$

Since $g \in \mathbb{E}^{m,\alpha}$, the solution X is then $C^{m+1,\alpha}$ up to ∂M. Hence the form $\kappa = \delta^*X$ is $C^{m-1,\alpha}$ up to ∂M and is an infinitesimal Einstein deformation in Bianchi gauge, i.e. $L(\kappa) = L_E(\kappa) = 0$ with $\beta(\kappa) = 0$. Note that by construction, $\kappa \in K$ so that $\kappa^T = 0$ at ∂M.

Next, note that

$$\mathcal{L}_X A = 2A'_\kappa. \tag{2.25}$$

Namely, since $\kappa = \frac{1}{2}\mathcal{L}_X g$, as in (2.11) one has $A'_\kappa = \frac{1}{4}\mathcal{L}_N\mathcal{L}_X g + \frac{1}{2}\mathcal{L}_{N'}g = \frac{1}{2}\mathcal{L}_X A + \frac{1}{4}\mathcal{L}_{[N,X]}g + \frac{1}{2}\mathcal{L}_{N'}g$. It is easy to verify that $2[N,X] = -N'$, which gives (2.25).

Theorem 2.2 and (2.23) imply that the form κ on M is pure gauge near ∂M, i.e. there exists a vector field Z defined in a neighborhood Ω of ∂M, with $Z = 0$ at ∂M, such that

$$\kappa = \delta^* Z \ \text{ on } \ \Omega. \tag{2.26}$$

Of course the vector fields X and Z can only differ by a Killing field in Ω.

It then follows from a basically standard analytic continuation argument in the interior of M, cf. [9, §.VI.6.3] for instance, that the vector field Z may be extended so that (2.26) holds on all of M. A detailed proof of this is also given in [3, Lemma 2.6]. This analytic continuation argument requires the topological hypothesis (1.2) to obtain a well-defined, (single-valued) vector field Z on M. Moreover, since ∂M is connected, the condition $Z = 0$ on ∂M remains valid in the analytic continuation.

Since one has $\beta\delta^* Z = 0$ on M with $Z = 0$ on ∂M, it follows from Lemma 2.3 that $Z = \kappa = 0$ on M. This implies that $\delta^* X = 0$ on M, i.e. X has been extended to a Killing field on (M, g). ■

If, in place of the condition $\pi_1(M, \partial M) = 0$, one assumes that (M, g) is embedded as a domain in a complete, simply connected Einstein manifold $(\hat{M}, \hat{g})$, then the same argument as above shows that the vector field $Y = X - Z$ is a Killing field on $\Omega \subset M \subset \hat{M}$. It then follows directly from analytic continuation, cf. [9, §VI.6.4], that Y extends uniquely to a Killing field on all of $\hat{M}$, which proves Theorem 1.1 in this case also.

Remark 2.5. We point out that Proposition 2.4 also shows the following. Let k be an infinitesimal Einstein deformation of (M, g) in Bianchi gauge, so $L(k) = 0$. If $\pi_1(M, \partial M) = 0$ and (2.12) holds, then $k = 0$. The proof is the same as above.

3. ELLIPTIC BOUNDARY VALUE PROBLEMS FOR THE EINSTEIN EQUATIONS

In this section, we consider elliptic boundary value problems for the Bianchi-gauged Einstein operator Φ in (2.15) and the Fredholm properties of the Dirichlet boundary map Π_D in (2.6).

Recall that the kernel of the linearized operator L in (2.16) forms the tangent space $T_g Z_C$, ($g = \widetilde{g}$ here), and by Lemma 2.3,

$$T_g Z_C = T_g \mathbb{E}_C, \tag{3.1}$$

so that the kernel also represents the space of (non-trivial) infinitesimal Einstein deformations in Bianchi gauge. The natural Dirichlet-type boundary conditions for Φ are

$$\beta_{\widetilde{g}}(g) = 0, \ \ g^T = \gamma \ \text{ at } \ \partial M. \tag{3.2}$$

However, contrary to first impressions, the operator Φ with boundary conditions (3.2) does not form a well-defined elliptic boundary value problem, (for g near $\widetilde{g}$). This is due to the well-known constraint equations, induced by the Gauss and Gauss-Codazzi equations on ∂M:

$$\delta(A - H\gamma) = -Ric_g(N, \cdot) = 0, \tag{3.3}$$

$$|A|^2 - H^2 + s_\gamma = s_g - 2Ric_g(N,N) = (n-1)\lambda. \tag{3.4}$$

Here H is the mean curvature of ∂M in M, while s denotes the scalar curvature.

As will be seen in §4, the momentum or vector constraint (3.3) is an important issue in the study of the isometry extension or rigidity results discussed in the Introduction. On the other hand, the

Hamiltonian or scalar constraint (3.4) is important in understanding the Fredholm properties of the boundary map Π_D in (2.6). Thus for $g \in \mathbb{E}^{m,\alpha}$, one has $A \in S_2^{m-1,\alpha}(\partial M)$ so that (3.4) implies that $s_\gamma \in C^{m-1,\alpha}(\partial M)$. However, the space of metrics $\gamma \in Met^{m,\alpha}(\partial M)$ for which $s_\gamma \in C^{m-1,\alpha}(\partial M)$ is of infinite codimension in $Met^{m,\alpha}(\partial M)$. It follows that the linearization of the boundary map Π_D has infinite dimensional cokernel, at least when $m < \infty$, and so Π_D is never Fredholm. Hence, the boundary conditions (3.2) for the operator Φ are not elliptic.

Remark 3.1. It is worthwhile to understand situations where the linearization $D\Pi_D$ has infinite dimensional kernel and cokernel, even in the C^∞ case. Let

$$K = K_g = Ker D_g \Pi_D. \tag{3.5}$$

Via the slice representation $Z_C = \mathbb{E}_C \subset \mathbb{E}$ at $\widetilde{g} = g$, K consists of forms κ such that

$$L(\kappa) = 0 \text{ and } \beta_g(\kappa) = 0 \text{ on } M, \text{ with } \kappa^T = 0 \text{ on } \partial M. \tag{3.6}$$

Consider then the intersection $K \cap Im\delta^*$. Let Y be a vector field at ∂M, (not necessarily tangent to ∂M), and extend Y to a vector field on M to be the unique solution to the equation $\beta(\delta^* Y) = 0$ with the given boundary value, cf. Lemma 2.3. Then $L(\delta^* Y) = 0$ and the boundary condition $k^T = (\delta^* Y)^T = 0$ is equivalent to the equation

$$(\delta^* Y^T)^T + \langle Y, N \rangle A = 0 \ \text{ at } \ \partial M. \tag{3.7}$$

In particular if δ_T^* is the restriction of δ^* to vector fields tangent to ∂M at ∂M, then $K \cap Im\delta_T^*$ is isomorphic to the space of Killing fields on $(\partial M, \gamma)$.

On the other hand, if ∂M is totally geodesic on some open set $U \subset \partial M$, i.e. $A = 0$ on U, then the system (3.7) has solutions of the form $Y = fN$, for *any* f with *supp* $f \subset U$, so that $K \cap Im\delta^*$ is infinite dimensional. Such vector fields Y are infinitesimal isometries *at*, (as opposed to *on*), ∂M, in that they preserve the metric γ on ∂M to first order. Of course in general such Y do not extend to a Killing field on (M, g); see also Remark 4.3 for further discussion and examples. This behavior is classically very well-known in the context of surfaces embedded in $\mathbb{R}^3$, cf. [16], [5].

A similar phenomenon holds for the cokernel. Thus, suppose $(\partial M, \gamma)$ is totally geodesic in a domain $U \subset \partial M$. Consider the linearization $s'_\gamma(h)$, for $h \in Im(D\Pi_D)$. By differentiating the scalar constraint (3.4) in the direction h, one sees that $s'_\gamma(h) = 0$ on U, for any such h. It follows that $ImD\Pi_D$ has infinite codimension, even in the C^∞ case, in such situations. The same argument and conclusion holds if $A = 0$ at just one point in ∂M.

Very little seems to be understood in characterizing the situations where K is finite dimensional or $K = 0$. Again, this is the case even in the classical setting of closed surfaces embedded in $\mathbb{R}^3$.

The discussion above implies there is no natural elliptic boundary value problem for the Einstein equations associated with Dirichlet boundary values. To obtain an elliptic problem, one needs to add either gauge-dependent terms or terms depending on the extrinsic geometry of ∂M in (M, g). To maintain a determined boundary value problem, one then has to subtract part of the intrinsic Dirichlet boundary data on ∂M.

There are several ways to carry this out in practice, but we will concentrate on the following situation. First, ellipticity of the Bianchi-gauged Einstein operator $\Phi = \Phi_{\widetilde{g}}$ with respect to given boundary conditions - near a given solution - depends only on the linearized operator, so we assume $g = \widetilde{g}$ is Einstein and study the linearized operator L from (2.16) at (M, g). As usual, let γ be the induced metric on ∂M.

Let B be a $C^{m,\alpha}$ symmetric bilinear form on ∂M such that

$$\tau_B = B - (tr_\gamma B)\gamma < 0, \tag{3.8}$$

is negative definite; all the statements to follow hold equally well if τ_B is positive definite. This condition is equivalent to the statement that the sum of any $(n-1)$-eigenvalues of B with respect to γ is positive. For the choice $B = A$, the 2^{nd} fundamental form, this is just the statement ∂M is $(n-1)$-convex in (M, g), cf. (3.23) below.

In place of prescribing the boundary metric g^T or its linearization h^T on ∂M, only h^T modulo B will be prescribed. Thus, let $\pi_B : T_\gamma Met^{m,\alpha}(\partial M) \to S_2^{m,\alpha}(\partial M)/B$, be the natural projection and set $\pi_B(h) = [h^T]_B$. In place of the second equation in (3.2), we impose

$$[h^T]_B = h_1. \tag{3.9}$$

For example, when B equals the boundary metric γ, one is prescribing the trace-free part of h^T, i.e. the tangent space of conformal classes on ∂M. Another natural choice is $B = A$, the 2^{nd} fundamental form of ∂M. In this case, for regularity purposes, one must work instead with a smooth approximation to A, since $A \in S_2^{m-1,\alpha}(\partial M)$, or with a C^∞ background (M, g).

The simplest gauge-dependent term one can add to (3.2) is the equation $h(N, N) = h_{00}$, where N is the unit normal with respect to g, while the simplest extrinsic geometric scalar is the linearization H'_h of the mean curvature of ∂M in (M, g) in the direction h. As shown in [3], ellipticity holds for either of these boundary conditions. We will use a slightly more general result, whose proof is a simple modification of the proof in [3].

Proposition 3.2. *Suppose $B \in S_2^{m,\alpha}$ satisfies* (3.8) *and suppose σ is any positive definite form in $S_2^{m,\alpha}(\partial M)$. Then the Bianchi-gauged linearized Einstein operator L in* (2.16) *with boundary conditions*

$$\beta(h) = 0, \quad [h^T]_B = h_1, \ \langle A'_h, \sigma\rangle = tr_\sigma A'_h = h_2 \ \text{ at } \ \partial M, \tag{3.10}$$

is an elliptic boundary value problem of Fredholm index 0.

Proof: The leading order symbol of $L = D\Phi$ is given by

$$\sigma(L) = -|\xi|^2 I, \tag{3.11}$$

where I is the $N \times N$ identity matrix, with $N = (n+2)(n+1)/2$ the dimension of the space of symmetric bilinear forms on $\mathbb{R}^{n+1}$. In the following, the subscript 0 represents the direction normal to ∂M in M, and Latin indices run from 1 to n. The positive roots of (3.8) are $i|\xi|$, with multiplicity N.

Writing $\xi = (z, \xi_i)$, the symbols of the leading order terms in the boundary operators are given by:

$$\begin{gathered}
-2izh_{0k} - 2i\sum \xi_j h_{jk} + i\xi_k trh = 0,\\
-2izh_{00} - 2i\sum \xi_k h_{0k} + iztrh = 0,\\
h^T = (\gamma')^T \ \ mod\, B,\\
h_{00} = \omega \ \text{ or } \ H'_h = \omega,
\end{gathered}$$

where h is an $N \times N$ matrix. Then ellipticity requires that the operator defined by the boundary symbols above has trivial kernel when z is set to the root $i|\xi|$. Carrying this out then gives the system

$$2|\xi|h_{0k} - 2i\sum \xi_j h_{jk} + i\xi_k trh = 0, \tag{3.12}$$

$$2|\xi|h_{00} - 2i\sum \xi_k h_{0k} - |\xi|trh = 0, \tag{3.13}$$

$$h_{kl} = \phi b_{kk}\delta_{kl}, \tag{3.14}$$

$$h_{00} = 0 \ \text{ or } \ H'_h = 0, \tag{3.15}$$

where without loss of generality we assume B is diagonal, with entries b_{kk}, and ϕ is an undetermined function.

Multiplying (3.12) by $i\xi_k$ and summing gives

$$2|\xi|i\sum\xi_k h_{0k} = 2i^2\xi_k^2 h_{kk} - i^2\xi_k^2 trh.$$

Substituting (3.13) on the term on the left above then gives

$$2|\xi|^2 h_{00} - |\xi|^2 trh = -2\sum\xi_k^2 h_{kk} + |\xi|^2 trh,$$

so that

$$|\xi|^2 h_{00} - |\xi|^2 trh = -\sum\xi_k^2 h_{kk} = -\phi\langle B(\xi),\xi\rangle.$$

Using the fact that $\sum h_{kk} = trh - h_{00}$, this is equivalent to

$$\phi\langle B(\xi),\xi\rangle = \phi|\xi|^2 trB.$$

Since $\tau_B = B - (trB)\gamma$ is assumed to be definite, it follows that $\phi = 0$ and hence $h^T = 0$.

Next, a simple computation from (2.11) shows that to leading order, $tr_\sigma A'_h = tr_\sigma(\nabla_N h - 2\delta^*(h(N)^T))$, which has symbol $iz\sigma^{ij}h_{ij} - 2i\sigma^{ij}\xi_i h_{0j}$. Setting this to 0 at the root $z = i|\xi|$ and using the fact that $h^T = 0$ gives

$$\sigma^{ij}\xi_i h_{0j} = 0. \tag{3.16}$$

Now (3.12) and $h^T = 0$ gives $2|\xi|h_{0j} + i\xi_j h_{00} = 0$. Multiplying the first term by $\sigma^{ij}\xi_i$ and summing over i, j gives 0 by (3.16), and hence $\sigma^{ij}\xi_i\xi_j h_{00} = 0$. Since $\sigma > 0$, it follows that $h_{00} = 0$ and hence by (3.12) again, $h_{0k} = 0$ for all k. This gives $h = 0$, and hence the boundary data (3.10) are elliptic.

To prove the operator L with boundary data (3.10) is of Fredholm index 0, one may continuously deform the boundary data through elliptic boundary values to self-adjoint boundary data, which clearly has index 0. This is done in detail for the case $\sigma = \gamma$ in [3] and the proof for general $\sigma > 0$ is identical. Thus we refer to [3] for details as needed. The result then follows from the homotopy invariance of the index.

∎

Given $\widetilde{g} \in \mathbb{E}^{m,\alpha}$, and B as in (3.8), let $Met_B^{m,\alpha}(\partial M) = Met^{m,\alpha}(\partial M)/B$ be the space of equivalence classes of $C^{m,\alpha}$ metrics on ∂M (mod B), with natural projection or quotient map

$$\pi_B : Met^{m,\alpha}(\partial M) \to Met_B^{m,\alpha}(\partial M).$$

It follows from Proposition 3.2 and Lemma 2.3 that the map

$$\widetilde{\Pi}_{B.\sigma} : \mathbb{E}_C \to Met_B^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \tag{3.17}$$

$$\widetilde{\Pi}_{B,\sigma}(g) = ([g^T]_B, tr_\sigma A),$$

is Fredholm, of index 0, for g near $\widetilde{g}$.

In analogy to (3.5), let

$$\widetilde{K}_{B,\sigma} = KerD\widetilde{\Pi}_{B,\sigma}, \tag{3.18}$$

where the derivative is taken at $g = \widetilde{g}$. In contrast to K in (3.5), $\widetilde{K}_{B,\sigma}$ is always finite dimensional. One might call an Einstein metric $g \in \mathbb{E}$ *non-degenerate*, (or (B,σ)-nondegenerate), if

$$\widetilde{K}_{B,\sigma} = 0, \tag{3.19}$$

for some B, σ. Thus, g is non-degenerate if and only if g is a regular point of the boundary map $\widetilde{\Pi}_{B,\sigma}$ in which case $\widetilde{\Pi}_{B,\sigma}$ is a local diffeomorphism near g.

Remark 3.3. It is worth pointing out that if (M, g) is strongly non-degenerate, in the sense that $K = 0$ in (3.5), then Theorem 1.1 is easy to prove and holds without the assumptions on H or on $\pi_1(M, \partial M)$. To see this, let ϕ_s be a local curve of $C^{m+1,\alpha}$ diffeomorphisms of $\bar{M}$ with $\phi_0 = id$ such that $\frac{d}{ds}\phi_s|_{s=0} = X$. If X is a Killing field on $(\partial M, \gamma)$, then

$$\phi_s^*\gamma = \gamma + O(s^2).$$

The curve $g_s = \phi_s^* g$ is a smooth curve in $\mathbb{E}$, and by construction, one has $[h] = [\frac{dg_s}{ds}] \in KerD\Pi_D$, for Π_D as in (2.6). One may then alter the diffeomorphisms ϕ_s by composition with diffeomorphisms $\psi_s \in \mathcal{D}_1^{m+1,\alpha}$ if necessary, so that $\kappa = \frac{d\psi_s^*(g_s)}{ds} \in K_g$, where $K = K_g$ is the kernel in (3.5) and $[h] = [\kappa]$. Thus

$$\kappa = \delta^* X',$$

where $X' = \frac{d(\phi_s \circ \psi_s)}{ds}$ is $C^{m+1,\alpha}$ smooth up to $\bar{M}$. Note that $X' = X$ at ∂M. If $K_g = 0$, then this gives

$$\delta^* X' = 0 \text{ on } M,$$

so that X' is a Killing field on (M, g). Thus, any Killing field on $(\partial M, \gamma)$ extends to a Killing field on (M, g), as claimed. The same result and proof hold in general, for any infinitesimal Einstein deformation preserving the boundary metric $(\partial M, \gamma)$.

It follows that if this general isometry extension property fails, then the Dirichlet boundary map Π_D in (3.5) is necessarily degenerate.

Remark 3.4. Although currently the cokernel of $D\Pi_D$ remains hard to understand, cf. Remark 3.1, it is not difficult to describe the cokernel of $D\widetilde{\Pi}_{B,\sigma}$. For simplicity, set $(B, \sigma) = (\gamma, \gamma)$ and let $\widetilde{\Pi}_{\gamma,\gamma} = \widetilde{\Pi}_H$. Then define

$$\widetilde{C} = \{((\mathcal{L}_N\kappa)^T_\gamma, N(H'_\kappa)) : \kappa \in \widetilde{K}_{\gamma,\gamma}\}, \tag{3.20}$$

so that $\widetilde{C}$ represents Neumann-type data associated with the Dirichlet data in (3.9).

Note that $\widetilde{C} \subset S^{m,\alpha}_\gamma(\partial M) \times C^{m-1,\alpha}(\partial M)$, where $S^{m,\alpha}_\gamma(\partial M) = T_g Met^{m,\alpha}_\gamma(\partial M) \simeq S^{m,\alpha}(\partial M)/\langle\gamma\rangle$. Namely, for $\kappa \in \widetilde{K}_{\gamma,\gamma}$, one has $L(\kappa) = 0$ on M together with the elliptic boundary conditions $\beta(\kappa) = 0$, $\kappa^T_\gamma = 0$, and $H'_\kappa = 0$ on ∂M. Since g is $C^{m,\alpha}$ up to ∂M, elliptic boundary regularity applied to this system gives $\kappa \in C^{m+1,\alpha}$, (cf. [6, 10]), so that $\mathcal{L}_N\kappa \in S^{m,\alpha}_2(\partial M)$ and $N(H'_\kappa) \in C^{m-1,\alpha}(\partial M)$.

It is then not difficult to prove (although we will not give the proof here), that the space $\widetilde{C}$ is a slice for $CokerD\widetilde{\Pi}_H$ in $S^{m,\alpha}_\gamma(\partial M) \times C^{m,\alpha}(\partial M)$, so that

$$S^{m,\alpha}_\gamma(\partial M) \times C^{m-1,\alpha}(\partial M) = ImD\widetilde{\Pi}_H \oplus \widetilde{C}. \tag{3.21}$$

By restricting to the first factor, it follows immediately from (3.21) that

$$S^{m,\alpha}_\gamma(\partial M) = ImD\Pi_0 \oplus \widetilde{S}, \tag{3.22}$$

where $\widetilde{S} = \{(\mathcal{L}_N\kappa)^T_\gamma : \kappa \in \widetilde{K}_H\}$ and Π_0 is defined by $\Pi_0 = \pi_\gamma \circ \Pi_D$.

One may use the diffeomorphism group to pass from the space $\mathbb{E}_C$ of Bianchi-gauged Einstein metrics to the full space $\mathbb{E}$, thus passing from $\widetilde{\Pi}_H$ to the more natural Dirichlet boundary map Π_D. In more detail, the image $\mathcal{V} = D\Pi_D(\mathbb{E}_C) \subset TMet^{m,\alpha}(\partial M)$ projects onto a space of finite codimension in $S^{m,\alpha}_\gamma(\partial M)$ by (3.22). The full image $D\Pi_D(\mathbb{E})$ then consists of the span $\langle\mathcal{V}, Im\delta^*\rangle$, where δ^* acts on all vector fields at ∂M, not necessarily tangent to ∂M. It is an interesting question to understand when the closure of this space is of finite codimension in $TMet^{m,\alpha}(\partial M)$. This corresponds roughly to Π_D being Fredholm.

One situation where this occurs is the following. Define $\partial M \subset M$ to be p-convex if the sum of any p eigenvalues of the second fundamental form A of ∂M in (M, g) is positive, cf. also [15] for example. Thus, ∂M is 1-convex if $A > 0$ is positive definite, while ∂M is n-convex if $H > 0$. It is easy to see that A is $(n-1)$-convex if and only if the form $H\gamma - A$ is positive definite,

$$H\gamma - A > 0. \tag{3.23}$$

This condition is equivalent to the local convexity of ∂M in (M, g) when $n = 2$, but becomes progressively weaker in higher dimensions.

Proposition 3.5. *If ∂M is $(n-1)$-convex, so that* (3.23) *holds, then the space*

$$\mathcal{V} = \overline{ImD\Pi_D},$$

is of finite codimension in $S_2^{m-1,\alpha}(\partial M)$, where the closure is taken in the $C^{m-1,\alpha}$ topology.

Proof: Recall from Proposition 3.2 that the operator L in (2.16) with boundary data

$$\beta(h) = 0,\ [h^T]_B = h_1,\ tr_\sigma A'_h = h_2, \tag{3.24}$$

is elliptic, of Fredholm index 0, provided σ is positive (or negative) definite and provided $\tau_B = B - (tr_\gamma B)\gamma \in S_2^{m,\alpha}(\partial M)$, is also negative definite. For $B = A$, by (3.23) one has the required definiteness, but there is a loss of one derivative in that $\tau \in S_2^{m-1,\alpha}(\partial M)$. Thus, let A_ε be a (C^∞) smoothing of A, ε-close to A in the $C^{m-1,\alpha}$ topology. Then the system $L(h) = 0$ with boundary data

$$\beta(h) = 0,\ [h^T]_{A_\varepsilon} = h_1,\ tr_\sigma A'_h = h_2, \tag{3.25}$$

is elliptic, of Fredholm index 0. The kernel and cokernel are of finite and equal dimensions.

Let π_{A_ε} denote the projection onto $S_2^{m,\alpha}(\partial M)/\langle A_\varepsilon\rangle = S_\varepsilon^{m,\alpha}(\partial M)$. Then the image of $\pi_{A_\varepsilon} \circ D\Pi_D$ is of finite codimension in $S_{A_\varepsilon}^{m,\alpha}(\partial M)$. The fiber $(\pi_{A_\varepsilon})^{-1}(0)$ consists of symmetric forms of the form fA_ε. Note that the forms fA are in $ImD\Pi_D$, when Π_D is extended to the domain $\mathbb{E}^{m-1,\alpha}(M)$, in that

$$fA = \delta^*(fN) \ \text{ at } \ \partial M,$$

where $\delta^*(fN)$ is extended to M to be in Bianchi gauge. Since the forms fA are $C\varepsilon$-close to fA_ε in the $C^{m-1,\alpha}$ topology, when $|f|_{C^{m-1,\alpha}} \leq C$, it follows, (by letting $\varepsilon \to 0$), that the closure of $ImD\Pi_D$ is of finite codimension in $S_2^{m-1,\alpha}(\partial M)$.

■

Remark 3.6. Consider hyperbolic 3-space $\mathbb{H}^3(-1)$ divided by translation along a geodesic, giving a hyperbolic metric g_{-1} on $D^2 \times S^1$. The metric g_{-1} has the simple form

$$g_{-1} = dr^2 + \sinh^2 r(d\theta_1)^2 + \cosh^2 r(d\theta_2)^2.$$

As in the example discussed in the Introduction, let σ be any smooth embedded closed curve in the hyperbolic plane $D^2 = \mathbb{H}^2(-1)$ surrounding the origin and let D be the disc bounded by σ. Let $M = \pi^{-1}(D^2) \simeq D^2 \times S^1$ with $\partial M = \pi^{-1}(\sigma)$, so that M is a solid torus with boundary a flat torus T^2.

It is easy to see that ∂M is convex in M whenever σ is convex in $\mathbb{H}^2(-1)$. However the flat torus boundary has two Killing fields, only one of which, (namely the vertical field tangent to θ_2), extends to a Killing field on M whenever the geodesic or mean curvature of σ in $\mathbb{H}^2(-1)$ is non-constant. Thus, isometry extension fails, even though ∂M is strictly convex - in contrast to the case of rigidity of convex surfaces in $\mathbb{R}^3$, cf. [16].

4. ISOMETRY EXTENSION AND THE DIVERGENCE CONSTRAINT

By Proposition 2.4, the basic issue for the isometry extension property is to understand when a Killing field on $(\partial M, \gamma)$ preserves the 2^{nd} fundamental form A of ∂M in M. We begin with the following identity on $(\partial M, \gamma)$, which holds on any closed oriented Riemannian manifold.

Proposition 4.1. *Let X be a Killing field on $(\partial M, \gamma)$. Suppose τ is a divergence-free symmetric bilinear form on $(\partial M, \gamma)$. Then*

$$\int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle dV_\gamma = -2 \int_{\partial M} \langle \delta' \tau, X \rangle dV_\gamma, \tag{4.1}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X and $\delta' = \frac{d}{ds}\delta_{\gamma+sh}$ is the variation of the divergence on $(\partial M, \gamma)$ in the direction $h \in S_2(\partial M)$.

Proof: Since the flow of X preserves γ, one has

$$\int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle dV_\gamma = - \int_{\partial M} \langle \tau, \mathcal{L}_X h \rangle dV_\gamma. \tag{4.2}$$

Next, setting $\gamma_s = \gamma + sh$, the divergence theorem applied to the 1-form $\tau(X)$ on ∂M gives

$$0 = \int_{\partial M} \delta_{\gamma_s}(\tau(X)) dV_{\gamma_s} = \int_{\partial M} \langle \delta_{\gamma_s} \tau, X \rangle dV_{\gamma_s} - \tfrac{1}{2} \int_{\partial M} \langle \tau, \mathcal{L}_X \gamma_s \rangle dV_{\gamma_s}, \tag{4.3}$$

where the second equality is a simple computation from the definitions; the inner products are with respect to γ_s. Taking the derivative with respect to s at $s = 0$ and using the facts that X is a Killing field on ∂M and $\delta\tau = 0$, it follows that

$$\int_{\partial M} \langle \delta' \tau, X \rangle dV - \tfrac{1}{2} \int_{\partial M} \langle \tau, \mathcal{L}_X h \rangle dV = 0.$$

Combining this with (4.2) then gives (4.1). ■

We now examine the right side of (4.1) in connection with the divergence constraint (3.3); of course (3.3) implies that the form

$$\tau_A \equiv \tau = A - H\gamma,$$

cf. (3.8), is divergence-free on ∂M.

We first discuss the general perspective. As discussed in §2, one may view the pair (γ, A) as Cauchy data for the Einstein equations (2.3) at ∂M. The data (γ, A) are then formally freely specifiable subject to the constraints (3.3)-(3.4). Let $\mathcal{T}$ be the space of pairs (γ, τ) with τ divergence-free with respect to γ; here $\gamma \in Met^{m,\alpha}(\partial M)$, $\tau \in S_2^{m-1,\alpha}(\partial M)$. The space $\mathcal{T}$ is naturally a vector bundle over $Met^{m,\alpha}(\partial M)$,

$$\pi : \mathcal{T} \to Met^{m,\alpha}(\partial M), \tag{4.4}$$

with π the projection on the first factor. Let also $\mathcal{F} \subset \mathcal{T}$ be the subset of pairs satisfying the scalar constraint equation (3.4). When expressed in terms of $\tau = A - H\gamma$, (3.4) is equivalent to

$$|\tau|^2 - \frac{1}{n-1}(tr\tau)^2 + s_\gamma = (n-1)\lambda.$$

Pairs $(\gamma, \tau) \in \mathcal{F}$ determine formal solutions of the Einstein equations near ∂M. More precisely, let (t, x^i) be geodesic boundary coordinates for (M, g), so that by the Gauss Lemma, the metric g has the form

$$g = dt^2 + g_t, \tag{4.5}$$

where $t(x) = dist_g(x, \partial M)$ and g_t is the induced metric on the level set $S(t)$ of t. Pulling back by the flow lines of ∇t, g_t may be viewed as a curve of metrics on ∂M, and one may formally expand g_t in its Taylor series:

$$g_t \sim \gamma - tA - \tfrac{1}{2}t^2\dot{A} + \cdots , \tag{4.6}$$

where $\dot{A} = \nabla_N A = -\nabla_T A$, $T = \nabla t = -N$. As noted above, the terms (γ, A) are freely specifiable, subject to the constraints (3.3)-(3.4). All the higher order terms in the expansion (4.6) are then determined by γ and A. To see this, one first uses the standard Riccati equation

$$\nabla_T A + A^2 + R_T = 0, \tag{4.7}$$

where $R_T(X,Y) = \langle R(X,T)T, Y\rangle$, cf. [11]. A standard formula gives $\nabla_T A = \mathcal{L}_T A - 2A^2$. Also, by the Gauss equation, the curvature term R_T may be expressed as

$$R_T = Ric^T - Ric_{int} + HA - A^2,$$

where $H = trA$, Ric_{int} is the intrinsic Ricci curvature of $S(t)$ and Ric^T is the tangential part, (tangent to $S(t)$), of the ambient Ricci curvature. Substituting in (4.7) gives

$$\ddot{g} = -2Ric^T + 2Ric_{int} + 4A^2 - 2HA. \tag{4.8}$$

For Einstein metrics satisfying (2.3), the right side of (4.8) involves only the first order t-derivatives of the metric g. Thus, repeated differentiation of (4.8) shows that all derivatives $g_{(k)} = \mathcal{L}_T^k g$ are determined at the boundary M by the Cauchy data (γ, A), so that (γ, A) determines the formal Taylor expansion of the curve g_t in (4.5) at $t = 0$.

The Cauchy-Kovalevsky theorem implies that if (γ, τ) are real-analytic forms on ∂M, then the formal series (4.6) converges to g_t, so that one obtains an actual Einstein metric g as in (4.5), defined in a neighborhood of ∂M. Of course, such metrics will not in general extend to globally defined Einstein metrics on M.

Now the right side of (4.1) is closely related to the linearization of the divergence constraint. Thus, if (γ_s, τ_s) is a curve in $Met^{m,\alpha}(\partial M) \times S_2^{m-1,\alpha}(\partial M)$ with tangent vector $(\gamma', \tau') = (h, \tau')$ at $s = 0$, then by the Gauss-Codazzi equation one has

$$\delta'(\tau) + \delta(\tau') = -(Ric(N, \cdot))', \tag{4.9}$$

where δ' is defined as in (4.1). If (γ_s, τ_s) is a curve in $\mathcal{T}$, then

$$\delta'(\tau) + \delta(\tau') = 0; \tag{4.10}$$

this is the linearized divergence constraint.

Lemma 4.2. *If the derivative $D\pi$ in* (4.4) *is surjective at (γ, τ), $\tau = A - H\gamma$, then*

$$\mathcal{L}_X A = 0 \text{ on } \partial M, \tag{4.11}$$

for any Killing field X on $(\partial M, \gamma)$. Conversely, if (4.11) *holds for all such Killing fields X, then $D\pi$ is surjective.*

Proof: This result follows easily from Proposition 4.1, with $\tau = A - H\gamma$. Thus, (4.10) gives $\delta'(\tau) = -\delta(\tau')$, for the variation δ' of δ in any direction $h \in T_\gamma Met(\partial M)$, for some τ'. Hence, (4.1) gives

$$\mathcal{F}(h) = \int_{\partial M} \langle \mathcal{L}_X \tau, h\rangle = -2\int_{\partial M} \langle \delta(\tau'), X\rangle = 2\int_{\partial M} \langle \tau', \delta^* X\rangle = 0, \tag{4.12}$$

since X is a Killing field on $(\partial M, \gamma)$. Since h is arbitrary, this implies that

$$\mathcal{L}_X \tau = 0,$$

on ∂M, and (4.11) follows by taking the trace of this equation. The same proof also gives the converse as well, using the splitting (4.13) below. ■

Thus, given $g \in \mathbb{E}$ and its corresponding 2nd fundamental form A, giving the pair (γ, A) at ∂M, a fundamental issue is whether $D\pi$ is surjective at (γ, A), i.e. whether the linearized divergence constraint (4.10) is solvable, for any variation h of γ on ∂M, (or for a space of variations dense in $S_2(\partial M)$ in the L^2 norm). One cannot expect that this holds at a general pair $(\gamma, \tau) \in \mathcal{T}$. Namely, for any compact manifold ∂M, one has

$$\Omega^1(\partial M) = Im\delta \oplus Ker\delta^*, \tag{4.13}$$

where Ω^1 is the space of $(C^{m-1,\alpha})$ 1-forms on ∂M. Thus, solvability at (γ, τ) in general requires that

$$\delta'(\tau) \in Im\delta = (Ker\delta^*)^{\perp}. \tag{4.14}$$

Of course $Ker\delta^*$ is exactly the space of Killing fields on $(\partial M, \gamma)$, and so this space serves as a potential obstruction space.

Obviously, π is locally surjective when $(\partial M, \gamma)$ has no Killing fields. On the other hand, it is easy to construct examples where $(\partial M, \gamma)$ does have Killing fields and π is not locally surjective.

Example 4.3. Let $(\partial M, \gamma)$ be a flat metric on the n-torus T^n; for example $\gamma = d\theta_1^2 + \cdots + d\theta_n^2$. Let $\tau = f(\theta_1)d\theta_2^2$, (for example). Then $\delta\tau = 0$, for any C^1 function $f(\theta_1)$. The pair (γ, τ) is in $\mathcal{T}$, and in fact in $\mathcal{F} \subset \mathcal{T}$. Letting X be the Killing field ∂_{θ_1}, one has $\mathcal{L}_X\tau \neq 0$ whenever f is non-constant, so that by the converse of Lemma 4.2, π is not locally surjective at such (γ, τ).

If (γ, τ) above are real-analytic, then $(\partial M, \gamma)$ is the boundary metric of an Einstein metric defined on a thickening $\partial M \times I$ of ∂M. Of course in general, such thickenings will not extend to Einstein metrics on a compact manifold bounding ∂M.

To obtain examples on compact manifolds, one may use the examples of $\mathbb{R}^2 \times S^1$, S^3 or $\mathbb{H}^3(-1)/\mathbb{Z}$ discussed in the Introduction and Remark 3.5. Here one has an infinite dimensional space of isometric embeddings of a flat torus in $\mathbb{R}^2 \times S^1$, S^3 or $\mathbb{H}^3(-1)/\mathbb{Z}$ for which Killing fields on the boundary do not extend to Killing fields of the ambient space.

Now clearly $D\pi$ is surjective onto $ImD\Pi_D$, since $ImD\Pi_D$ consists of variations of the boundary metric determined by global variations of the Einstein metric g on M which of course satisfy (4.10). Hence if $D\Pi_D$ is onto, or has dense range in $S_2(\partial M)$, then Lemma 4.2 holds, i.e. (4.11) holds; compare with Remark 3.3. On the other hand, the examples above show that whether (4.11) holds or not must depend either on global properties of (M, g) or extrinsic properties of $\partial M \subset M$.

Next, we place the discussion above in a broader context of rigidity issues. The boundary $(\partial M, \gamma)$ of the Einstein manifold (M, g) is called infinitesimally (Einstein) rigid if the kernel K of $D\Pi_D$ in (3.5) is trivial, i.e. $K = 0$. Thus, infinitesimal rigidity is equivalent to the injectivity of $D\Pi_D$. It is also equivalent to the local rigidity of $(\partial M, \gamma)$, (i.e. the local uniqueness of an Einstein filling (M, g) up to isometry), by the manifold theorem, Theorem 2.1.

For example, suppose X is an infinitesimal isometry at $(\partial M, \gamma)$, in that $(\delta^*X)^T = 0$ at ∂M, (X is not necessarily tangent to ∂M). Then as discussed in Remark 3.1, the deformation δ^*X may be extended uniquely to M by choosing it to be in Bianchi gauge. Then $\delta^*X \in K$ and infinitesimal rigidity of ∂M implies that $k = 0$, so that X is a Killing field on (M, g). Rigidity in this more restricted sense will be called infinitesimal isometric rigidity. Both forms of such rigidity are of course generalizations of the isometry extension property discussed in the Introduction.

One may obtain analogs of Proposition 4.1 and Lemma 4.2 in this context via the Einstein-Hilbert action. Thus, recall that Einstein-Hilbert action with Gibbons-Hawking-York boundary term on M is

$$I(g) = I_{EH}(g) = -\int_M (s_g - 2\Lambda)dV_g - 2\int_{\partial M} H dv_\gamma, \tag{4.15}$$

where $\Lambda = \frac{n-1}{2}\lambda$, cf. [7]. The 1st variation of I in the direction h is given by

$$\frac{d}{dr}I(g+rh) = \int_M \langle \hat{E}_g, h\rangle dV_g + \int_{\partial M} \langle \tau_g, h\rangle dv_\gamma, \tag{4.16}$$

where $\hat{E}$ is the Einstein tensor, $\hat{E}_g = Ric_g - \frac{s}{2}g + \Lambda g$ and $\tau = A - H\gamma$ is as above. Here and below, all parameter derivatives are taken at 0. Einstein metrics with $Ric_g - \lambda g = 0$ are critical points of I, among variations vanishing on ∂M. Consider a 2-parameter family of metrics $g_{r,s} = g + rh + sk$ where $E_g = 0$. Then

$$\frac{d^2}{dsdr}I(g_{r,s}) = \frac{d^2}{drds}I(g_{r,s}). \tag{4.17}$$

Computing the left side of (4.17) by taking the derivative of (4.16) in the direction k gives

$$\frac{d^2}{dsdr}I(g_{r,s}) = \int_M \langle \hat{E}'(k), h\rangle dV_g + \int_{\partial M} \langle \tau'_k + a(k^T), h^T\rangle dv_\gamma. \tag{4.18}$$

Since $\hat{E}_g = 0$, there are no further derivatives of the bulk integral in (4.16). Also, $a(k) = -2\tau \circ k + \frac{1}{2}(tr_\gamma k)\tau$ arises from the variation of the metric and volume form in the direction k; by definition $(\tau \circ k)(V,W) = \frac{1}{2}\{\langle \tau(V), k(W)\rangle + \langle \tau(W), k(V)\rangle\}$.

Similarly, for the right side of (4.17) one has

$$\frac{d^2}{drds}I(g_{r,s}) = \int_M \langle \hat{E}'(h), k\rangle dV_g + \int_{\partial M} \langle \tau'_h + a(h^T), k^T\rangle dv_\gamma. \tag{4.19}$$

In particular, suppose k_D is an infinitesimal Einstein deformation in the kernel K from (3.5), so that $k_D|_{\partial M} = k^T = 0$. If $h \in T\mathbb{E}$ is any infinitesimal Einstein deformation, then (4.17)-(4.19) gives,

$$\int_{\partial M} \langle \tau'_{k_D}, h\rangle dv_\gamma = \int_{\partial M} \langle \tau'_h, k_D\rangle dv_\gamma = 0. \tag{4.20}$$

One thus has

$$I''(k_D, h) = 0,$$

on-shell. Note this computation recaptures (4.12) when $k_D = \delta^* X = \frac{1}{2}\mathcal{L}_X g$.

Proposition 4.4. *If $\pi_1(M, \partial M) = 0$ and the linearization $D\Pi_D$ has dense range in $S_2^{m,\alpha}(\partial M)$, then $D\Pi_D$ is injective, so that $K = 0$ in (3.5) and $(\partial M, \gamma)$ is infinitesimally Einstein rigid.*

Proof: The proof is a simple consequence of (4.16)-(4.18) and Theorem 2.2. Thus, suppose $k \in K$ so that k is an infinitesimal Einstein deformation with $k^T = 0$ at ∂M. By (4.20),

$$\int_{\partial M} \langle \tau'_k, h\rangle = \int_{\partial M} \langle \tau'_h, k\rangle = 0, \tag{4.21}$$

for any $h \in ImD\Pi_D$. Since $ImD\Pi_D$ is dense in $S_2^{m,\alpha}(\partial M)$, it follows that $(\tau'_k)^T = 0$ on ∂M. Taking the trace, it follows that

$$k^T = 0 \ \text{ and } \ (A'_k)^T = 0 \ \text{ on } \ \partial M.$$

Hence by Theorem 2.2 and the same argument as in the proof of Proposition 2.4, it follows that $k = 0$ on M, which proves the result.

■

It is an open question whether converse holds, i.e. if the injectivity of $D\Pi_D$ implies $D\Pi_D$ has dense range. By the discussion in §3, $D\Pi_D$ is never surjective onto $S_2^{m,\alpha}(\partial M)$, when $m < \infty$.

Remark 4.5. There are simple examples of Einstein metrics which are not infinitesimally rigid, even when ∂M is convex. Perhaps the simplest example is given by the curve of Riemannian Schwarzschild metrics g_m on $\mathbb{R}^2 \times S^2$, given by

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{S^2(1)},$$

where $V = V(r) = 1 - \frac{2m}{r}$, $r \geq 2m > 0$. Smoothness at the horizon $\{r = 2m\}$ requires that $\theta \in [0, \beta]$ where $\beta = 8\pi m$, so that g_m may be rewritten in the form

$$g_m = V^{-1}dr^2 + 64\pi^2 m^2 V d\theta^2 + r^2 g_{S^2(1)}, \tag{4.22}$$

where now $\theta \in [0, 1]$. This is a curve of complete Ricci-flat metrics, but the metrics g_m differ from each other just by rescalings and diffeomorphisms. Taking the derivative with respect to m gives an infinitesimal Einstein deformation κ of g_m:

$$\kappa = 2m64\pi^2[1 - \frac{3m}{r}]d\theta^2 + \frac{2}{r}(1 - \frac{2m}{r})^{-2}dr^2. \tag{4.23}$$

For the moment, fix $m > 0$ and let $M = M(R) = \{2m \leq r \leq R\}$. The restriction of g_m to M gives a curve of Einstein metrics on the bounded domain $D^2 \times S^2$ with boundary $\partial M \simeq S^1 \times S^2$ and boundary metric

$$\gamma = \gamma_R = 64\pi^2 m^2[1 - \frac{2m}{R}]d\theta^2 + R^2 g_{S^2(1)}.$$

Let $\omega(R)$ be the ratio of the radii of the S^1 and S^2 factors at ∂M, so that

$$\omega(R) = \frac{64\pi^2 m^2[1 - \frac{2m}{R}]}{R^2}.$$

Then $\omega(R) \to 0$ as $R \to 0$ and $R \to \infty$, and has a single maximum value $64\pi^2/27$ at the critical point $R = 3m$ where $\kappa^T = 0$. At this critical radius, equal to the photon radius of the Lorentzian Schwarzschild metric, the boundary metric has the form

$$\gamma = \frac{64}{3}\pi^2 m^2 d\theta^2 + 9m^2 g_{S^2(1)},$$

and a simple calculation shows that the 2^{nd} fundamental form A is umbilic, with

$$A = \frac{1}{3\sqrt{3}m}\gamma.$$

The discussion above shows that the Einstein metric g_m is not infinitesimally rigid on the domain $M(3m)$; the form κ in (4.23) is in $KerD\Pi_D$. Proposition 4.4 implies that $D\Pi_D$ does not have dense range on $M(3m)$; in fact boundary metrics for which the mass-independent ratio $\omega > \omega_0 = 64\pi^2/27$ are not in $Im\Pi_D$, (at least along the Schwarzschild curve). The Dirichlet boundary map Π_D has a simple fold behavior near the critical radius, and so has local degree 0. It is shown in [17] that the Schwarzschild metric g_m on $M(R)$ is stable, in that the 2^{nd} variation of the action (4.15) is positive definite, for $R < 3m$, while it becomes unstable, (has a negative mode or eigenvalue), when $R > 3m$.

A detailed discussion of the physical aspects of the Schwarzschild curve is given in [17], and further examples in both four and higher dimensions are discussed in [1] and references therein.

A simple computation using (2.11) shows that on the domain $M(3m)$

$$A'_\kappa = \frac{1}{\sqrt{3}m^2}(\theta^1)^2 - \frac{1}{3\sqrt{3}m^2}\gamma,$$

where θ^1 is the unit 1-form in the direction θ. This shows that $H'_\kappa = 0$ at ∂M. Hence, the form κ is also in the kernel of the Fredholm boundary map $\widetilde{\Pi}_{B,\gamma}$ in (3.17). This shows that the generalization of Theorem 1.1 to infinitesimal Einstein rigidity is false; the form κ is a non-trival infinitesimal Einstein deformation preserving the boundary metric and mean curvature. Of course κ is not of the form $\delta^* X$ for some vector field X.

5. Proof of the Main Results.

In this section, we prove the main results discussed in the Introduction, beginning with Theorem 1.1. As noted above, one needs to use global arguments to prove Theorem 1.1. We do this by studying global properties of the linearized operator L from (2.16).

Consider the elliptic boundary value problem:

$$L(h) = \ell, \ \text{ on } M, \ \ \beta(h) = h_0, \ [h^T]_0 = h_1, \ H'_h = h_2 \text{ on } \partial M, \tag{5.1}$$

where $[h^T]_0 = [h^T]\gamma$, (the trace-free part of h^T). By Proposition 3.2, this is an elliptic boundary value problem of Fredholm index 0. Let K denote the kernel, so that $k \in K$ means

$$L(k) = 0, \ \ \beta(k) = 0, \ [k^T]_0 = 0, \ H'_k = 0.$$

If $K = 0$, then L is surjective and so the form ℓ and boundary values for $\beta(h), [h^T]_0$ and H'_h may be freely chosen; given arbitrary ℓ and h_i, $0 \leq i \leq 2$, the system (5.1) has a unique solution, (when suitable smoothness assumptions are imposed).

Now as in Lemma 4.2, (using (4.10) and (4.12)), one has

$$\int_{\partial M} \langle \mathcal{L}_X \tau, h^T \rangle = \int_{\partial M} (Ric(N,X))'_h. \tag{5.2}$$

We will prove that any deformation h^T of γ on ∂M extends to a deformation h of g on M such that the right side of (5.2) vanishes; Theorem 1.1 then follows easily.

Note first that (5.2) vanishes in pure-trace directions $h^T = f\gamma$. Namely, since X is Killing, $tr(\mathcal{L}_X \tau) = -(n-1)X(H) = 0$, by assumption. Hence, $\langle \mathcal{L}_X \tau, f\gamma \rangle = 0$ pointwise and so the right side of (5.2) vanishes in pure-trace directions also.

By Lemma 2.3, deformations h satisfying

$$L(h) = 0, \ \text{ on } M, \ \ \beta(h) = 0 \ \text{ on } \partial M, \tag{5.3}$$

are infinitesimal Einstein deformations in Bianchi gauge on M and hence, at ∂M,

$$(Ric(N,X))'_h = 0,$$

since N is normal and X is tangential. Now write any h^T on ∂M as $h^T = h_0 + f\gamma$ where h_0 is trace-free. Let $\bar{f}$ be any smooth function and let $\bar{h}^T = h_0 + \bar{f}\gamma$, so that $\bar{h}^T - h^T$ is pure-trace. Then by the remarks following (5.2)

$$\int_{\partial M} (Ric(N,X))'_{\bar{h}} = \int_{\partial M} (Ric(N,X))'_h. \tag{5.4}$$

Suppose first the boundary value problem in (5.1) has trivial kernel, $K = 0$. It follows that there exists an infinitesimal Einstein deformation h of (M, g) satisfying (5.3) with $h^T = h_0 + f\gamma$, for some f and with the class $[h^T]_0 = h_1$ arbitrarily prescribed. For all such h, it follows that

$$\int_{\partial M} (Ric(N,X))'_h = 0. \tag{5.5}$$

Via (5.4), (5.5) then also holds for all h, and so by (5.2), one obtains

$$\mathcal{L}_X \tau = 0.$$

Since $tr(\mathcal{L}_X \tau) = 0$, this gives $\mathcal{L}_X A = 0$ and Theorem 1.1 then follows from Proposition 2.4.

Next, suppose $K \neq 0$. We consider small Fredholm perturbations of L which are non-singular, i.e. have no kernel. Thus, let Q be a smooth symmetric bilinear form on M and consider the operator

$$\widetilde{L}(h) = L(h) + \langle h, Q\rangle g, \tag{5.6}$$

with the same boundary values as (5.1). Given any $\varepsilon > 0$, it is clear that there exist Q such that $|Q| \leq \varepsilon$ smoothly and such that

$$\widetilde{L}(k) \neq 0,$$

for all $k \neq 0$ in K. This is a simple consequence of the fact that K is finite dimensional. Let $K^{\perp}$ be the L^2 orthogonal complement of K within the space of forms h with 0 boundary values in (5.1). The operator $L|_{K^{\perp}}$ is then an isomorphism onto its image; the norm of $L_{K^{\perp}}$ as well as the norm of the inverse operator, are uniformly bounded. Hence, for suitable Q sufficiently small, the operator $\widetilde{L}$ in (5.6) with boundary data (5.1) is Fredholm, of index 0, and has trivial kernel.

Consider then the boundary value problem

$$\widetilde{L}(h) = 0, \;\; \beta(h) = d\alpha, \;\; [h^T]_0 = h_1, \;\; H'_h = h_2. \tag{5.7}$$

This has a unique solution, for arbitrary α, h_1 and h_2. Of course solutions of (5.7) are not infinitesimal Einstein deformations in general, even when $\alpha = 0$. However, as shown below, this difference can be controlled, cf. (5.21).

The Bianchi identity gives $\beta(E) = 0$, where E is the Einstein operator in (2.7). Hence, $\beta E' = 0$, at any Einstein metric and so by (2.17), $\beta(L(h)) = 2\beta\delta^*(\beta(h))$. It follows from (5.6)-(5.7) that if h is a solution of (5.7), then

$$0 = \beta\widetilde{L}(h) = 2\beta\delta^*(\beta(h)) + \tfrac{n-1}{2}d\langle h, Q\rangle. \tag{5.8}$$

The solution $\beta(h)$ of (5.8) is uniquely determined by the 0-order term $\chi = \chi(h) = \frac{n-1}{2}\langle h, Q\rangle$, given the boundary value $d\alpha$ on ∂M.

Claim. For any given h_1 in (5.7), there exists α on ∂M, $\alpha' \equiv d\alpha(N)$ and h_2 at ∂M, such that the solution h of (5.7) satisfies

$$\beta(h) = d\bar{\alpha}, \tag{5.9}$$

for some function $\bar{\alpha}$ on M.

To prove this claim, observe first that a standard Weitzenbock formula gives $2\beta\delta^* V = d\delta V + \delta dV - 2Ric(V)$, so that (5.8) is equivalent to

$$d\delta V + \delta dV - 2\lambda V + d\chi = 0, \tag{5.10}$$

with $V = \beta(h)$. Let $\bar{\alpha}$ be the solution of the equation

$$\Delta\bar{\alpha} + 2\lambda\bar{\alpha} = \chi, \tag{5.11}$$

inducing a given boundary function α on ∂M; the equation (5.11) is uniquely solvable when $\lambda \leq 0$. Substituting this in (5.10) and setting $W = V - d\bar{\alpha}$ gives

$$d\delta W + \delta dW - 2\lambda W = 0, \tag{5.12}$$

with $W^T = 0$ on ∂M. The normal component $W^N = \langle W, N\rangle$ is given by

$$\langle W, N\rangle = \alpha' - N(\bar{\alpha}), \tag{5.13}$$

where $N(\bar{\alpha})$ is the Neumann boundary data of the solution of (5.11). If $\langle W, N\rangle = 0$, so that $W = 0$ at ∂M, then we claim that $W = 0$ on M. To see this, pair (5.12) with W and integrate by parts to obtain

$$\int_M |dW|^2 + (\delta W)^2 - 2\lambda|W|^2 - \int_{\partial M} (dW)(W, N) - (\delta W)\langle W, N\rangle = 0. \tag{5.14}$$

If $\lambda < 0$ this implies $W = 0$. If $\lambda = 0$, then $dW = \delta W = 0$ and hence again by a standard Weitzenbock formula, $D^*DW = 0$. Pairing this with W and integrating by parts implies that $DW = 0$ so that W is parallel, and hence, (since $W = 0$ on ∂M), $W = 0$.

Now the claim above follows from the statement that, given arbitrary but fixed boundary data h_1 in (5.7), there exists $\alpha \in C^{k,\beta}(\partial M)$ and $\alpha' = (\beta(h))(N) \in C^{k-1,\beta}(\partial M)$ such that $\langle W, N\rangle = 0$, (for some value of h_2). Observe that the term χ in (5.6) and (5.11) depends on the solution h of (5.7), as does W itself via $\beta(h)$. To prove the claim, consider then the smooth map

$$F_{(h_1,h_2)} : C^{k,\beta}(\partial M) \times C^{k-1,\beta}(\partial M) \to C^{k-1,\beta}(\partial M), \tag{5.15}$$

$$F_{(h_1,h_2)}(\alpha, \alpha') = N(\bar{\alpha}).$$

Namely, since the data h_1 and h_2 are fixed, the pair (α, α') uniquely define the solution h in (5.7) and thus the Neumann boundary value $N(\bar{\alpha})$ via (5.11). Note that if $h_1 = h_2 = 0$ in (5.7) then the solution h of (5.7) is bilinear in (α, α'), and hence so is $\chi = \chi(h)$. It follows that F is bilinear in this situation, and hence F is bi-affine in α, α' in general. Of course the map F depends on the choice of Q in (5.6).

Suppose first that $Q = 0$, so that $\chi = 0$. In this case, the map $F = F_0$ is linear and independent of h and of α'; it is just the Dirichlet-to-Neumann map for the operator $\Delta + 2\lambda$ in (5.11). Given any α on ∂M, one may then just choose α' as $\alpha' = N(\bar{\alpha})$, which then gives $\langle W, N\rangle = 0$, as desired.

Next, write $(\alpha, \alpha', h_1, h_2) = (\alpha, 0, h_1, h_2) + (0, \alpha', 0, 0)$ and suppose that the map $(0, \alpha', 0, 0) \to h$ satisfies

$$|h| \leq C(|\alpha'|), \tag{5.16}$$

for some fixed constant C independent of ε, (see the discussion following (5.6)). Here the norms are the $C^{k,\beta}$ norms, as in (5.15). Since $|Q| \leq \varepsilon$, it follows that $|\chi| \leq \varepsilon C(|\alpha'|)$, and so the map $F_{(0,0)}(0, \alpha', 0, 0) = N(\bar{\alpha})$ mapping $C^{k-1,\beta} \to C^{k-1,\beta}$ is an ε'-contracting map, for $\varepsilon' = \varepsilon'(\varepsilon, C, M)$.

To use this property, fix any α_0, and let $N(\bar{\alpha}_0) = F_{(h_1,h_2)}(\alpha_0, 0)$. Then $F_{(h_1,h_2)}(\alpha_0, \alpha') = F_{(h_1,h_2)}(\alpha_0, 0) + F_{(0,0)}(0, \alpha') \equiv N(\bar{\alpha}_0) + N(\bar{\alpha}) = N(\bar{\alpha}_0 + \bar{\alpha})$. Hence it suffices to prove that there exists α' such that

$$F_{(\alpha_0,h_1,h_2)}(\alpha') = N(\bar{\alpha}_0 + \bar{\alpha}) = \alpha', \tag{5.17}$$

i.e. it suffices to find a fixed point for $F_{(\alpha_0,h_1,h_2)}$. But the discussion following (5.16) implies that $F_{(\alpha_0,h_1,h_2)}$ is an ε'-contracting map, and hence has a unique fixed point, (for ε sufficiently small), which completes the proof in this case. Note this argument holds for α_0 arbitrary.

Now in the situation at hand, (5.16) does not hold, due exactly to the presence of the kernel K preceding (5.6). The equation $L(h) = 0$ is solvable for boundary values $(\alpha, \alpha', h_1, h_2)$ in an image space $\mathcal{I}$ of finite codimension in the product. The projection of $\mathcal{I}$ onto each factor is a space $\mathcal{I}_i$, $1 \leq i \leq 4$ of finite codimension in each factor. The map $K^{\perp} \to \mathcal{I}$, taking h to its four components of boundary data is an isomorphism, with bounded norm and with inverse of bounded norm. This estimate also holds for solutions of the perturbed equation $\widetilde{L}(h) = 0$ on $K^{\perp}$, with bound independent of ε, (as in (5.16)), but the boundary map on K has inverse with large norm, (on the order of ε^{-1}).

Let $\mathcal{I}_2$ be the corresponding subspace of the 2nd factor, (i.e. α'). We claim that for any h_1, there exists $\hat{\alpha}$, $\hat{\alpha}'$ and h_2, (depending on h_1), such that $F_{(h_1,h_2)}(\hat{\alpha}, \hat{\alpha}') \in \mathcal{I}_2$. Given this, the same argument as above in (5.17) applied to $(0, \alpha', 0, 0)$ with α' restricted to the space $\mathcal{I}_2$ will give the existence of the required fixed point

$$F_{(h_1,h_2)}(\hat{\alpha}, \alpha') = \alpha'.$$

To do this, consider pure-trace solutions of (5.7). Thus, let $h = \phi g$. Then $\widetilde{L}(h) = 0$ if and only if

$$\Delta\phi + 2\lambda\phi - \phi trQ = 0. \tag{5.18}$$

One may specify the Neumann boundary value $N(\phi)$ arbitrarily, (for a generic choice of Q), which then determines the Dirichlet boundary value ϕ on ∂M. For such h, one has $[h^T]_0 = 0$, so that $h_1 = 0$; (the term h_2 is determined by $N(\bar{\alpha})$ here). Also $\chi = \frac{n-1}{2}\phi trQ$, so that (5.11) becomes

$$\Delta\bar{\alpha} + 2\lambda\bar{\alpha} - \tfrac{n-1}{2}\phi trQ = 0. \tag{5.19}$$

Comparing (5.18) and (5.19), one may thus set

$$\bar{\alpha} = \tfrac{n-1}{2}\phi, \tag{5.20}$$

with $N(\bar{\alpha})$ arbitrarily given on ∂M; in particular one may choose $N(\bar{\alpha}) \in \mathcal{I}_2$. This then determines α on ∂M. One can then add to such a choice arbitrary data $(0, \alpha', 0, h_2)$, with α' running over $\mathcal{I}_2$, and apply the contraction mapping argument above to obtain a fixed point, as in (5.17). This proves the claim (5.9).

Writing out the equation for $\widetilde{L}$ in (5.7) and using (5.9) with (2.16)-(2.17), it then follows that the equation

$$2E'_h + 2\delta^* d\bar{\alpha} + \tfrac{2}{n-1}\chi g = 0, \tag{5.21}$$

has solutions with $[h^T]_0 = h_1$ arbitrarily prescribed on ∂M. One has $E'_h(N, X) = (Ric(N, X))'_h$, and so (5.21) implies that

$$\int_{\partial M} (Ric(N, X))'_h = -\int_{\partial M} D^2\bar{\alpha}(N, X) = -\int_{\partial M} \langle \nabla_X d\bar{\alpha}, N\rangle. \tag{5.22}$$

Now $\int_{\partial M} \langle \nabla_X d\bar{\alpha}, N\rangle = \int_{\partial M} XN(\bar{\alpha}) - \langle d\bar{\alpha}, A(X)\rangle$. The first term here vanishes since X is divergence free, while via integrations by parts, the second term equals $\int_{\partial M} \bar{\alpha}\delta(A(X)) = \int_{\partial M} \bar{\alpha}[(\delta A)(X) + \langle A, \delta^* X\rangle] = \int_{\partial M} \bar{\alpha}(\delta A)(X)$, since X is Killing on ∂M. Finally, by the constraint equation, $(\delta A)(X) = dH(X) = X(H) = 0$, by assumption. Hence,

$$\int_{\partial M} (Ric(N, X))'_h = 0. \tag{5.23}$$

This holds for $[h^T]_0$ arbitrary, and so as in (5.4), (5.23) holds for all variations h on ∂M. As above via (5.2) and Proposition 2.4, this completes the proof.

■

Remark 5.1. We expect Theorem 1.1 (and Corollary 1.2) hold also when $\lambda > 0$. The proof above is valid in this situation when there are no non-zero solutions of (5.11) with $\chi = 0$ and (5.12) with zero boundary values.

Remark 5.2. The proof of Theorem 1.1 above shows that, when for instance $H = const$ at ∂M, one has

$$K \cap Im\delta^* = 0, \tag{5.24}$$

where δ^* acts on vector fields X tangent to ∂M, and $K = KerD\Pi_D$, as in (3.5).

However, for instance in dimension 3, all Einstein deformations are pure gauge, i.e. of the form $\delta^* V$, for some vector field V, not necessarily tangent to ∂M, cf. Remark 3.1. Hence, if Π_D is degenerate at some constant curvature metric (M^3, g), i.e. $K = K_g \neq 0$ and again $H = const$ at ∂M, then

$$K \cap \delta^* V \neq 0,$$

for general V at ∂M. The condition $H = const$ is necessary here, cf. Example 4.3.

The proof of Theorem 1.1 above generalizes to the case of isometric rigidity, where the vector field X is not assumed tangent to ∂M, but is a general vector field at ∂M satisfying the conditions

$$(\delta^* X)^T = 0, \quad \text{and} \quad tr_\gamma A'_{\delta^* X} = 0, \tag{5.25}$$

in place of $X(H) = 0$.

Proposition 5.3. *Let X be a vector field at ∂M generating an infinitesimal isometry at ∂M and suppose* (5.25) *holds. If $\lambda \leq 0$ and $\pi_1(M, \partial M) = 0$, then X extends to a Killing field on (M, g).*

Proof: The proof is a relatively simple extension of the proof of Theorem 1.1. First note that (5.21) holds in general, where we recall $E = Ric_g - \lambda g$. Using (4.18)-(4.19) together with (5.25) one obtains, for any deformation h of g,

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} \langle \tau'_h, (\delta^* X)^T \rangle + \int_M \langle \hat{E}'_h, \delta^* X \rangle = \int_M \langle \hat{E}'_h, \delta^* X \rangle,$$

where $\hat{E} = Ric - \frac{s}{2} g + \Lambda g$. Integrating the right-hand side by parts, and using the fact that $\delta \hat{E}' = 0$, it follows that

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} \hat{E}'_h(X, N).$$

Now $\hat{E}'_h = (Ric - \lambda g)'_h + \lambda h - \frac{s'_h}{2} g - \frac{s}{2} h + \Lambda h = (Ric - \lambda g)'_h - \frac{s'_h}{2} g$, since g is Einstein. This gives

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} E'_h(X, N) - \frac{s'_h}{2} \langle X, N \rangle,$$

and $s'_h = -\Delta trh + \delta\delta h - \langle Ric_g, h \rangle$.

Now we may assume that h is a solution of (5.21) with $\beta(h)$ satisfying (5.9). Then $\delta h + \frac{1}{2} dtrh = d\bar{\alpha}$, so that $\delta\delta h = \frac{1}{2} \Delta_M trh - \Delta_M \bar{\alpha}$. Hence, $s'_h = -\frac{1}{2} \Delta_M trh - \lambda trh - \Delta_M \bar{\alpha}$. Also, taking the trace of (5.7) gives $-\Delta_M trh - 2\lambda trh + (n+1) \langle h, Q \rangle = 0$, so that $s'_h = -\frac{n+1}{2} \langle h, Q \rangle - \Delta_M \bar{\alpha}$. Thus

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} E'_h(X, N) + [\tfrac{n+1}{4} \langle h, Q \rangle + \tfrac{1}{2} \Delta_M \bar{\alpha}] \langle X, N \rangle. \tag{5.26}$$

Now we compute the right-hand side of (5.26) using (5.21). Writing $X = \omega N + X^T$ at ∂M, one then has

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} -\langle \nabla_X d\bar{\alpha}, N \rangle - \omega \tfrac{1}{n-1} \chi + \omega \tfrac{n+1}{4} \langle h, Q \rangle + \tfrac{1}{2} \omega \Delta_M \bar{\alpha}.$$

Since $\chi = \frac{n-1}{2} \langle h, Q \rangle$, this gives

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h \rangle = \int_{\partial M} -\langle \nabla_X d\bar{\alpha}, N \rangle + \omega \tfrac{n-1}{4} \langle h, Q \rangle + \tfrac{1}{2} \omega \Delta_M \bar{\alpha}. \tag{5.27}$$

Next,

$$\langle \nabla_X d\bar{\alpha}, N \rangle = \omega NN(\bar{\alpha}) + X^T N(\bar{\alpha}) - \langle d\bar{\alpha}, A(X^T) \rangle. \tag{5.28}$$

For the first term, one has $NN(\bar{\alpha}) = \Delta_M \bar{\alpha} - \Delta_{\partial M} \bar{\alpha} - HN(\bar{\alpha})$. Also

$$\int_{\partial M} X^T N(\bar{\alpha}) = -\int_{\partial M} div(X^T) N(\bar{\alpha})$$

and $div(X^T) + H\omega = tr_{\partial M} (\delta^* X)^T = 0$. Substituting these calculations into (5.28) gives a cancelation leaving

$$\int_{\partial M} \langle \nabla_X d\bar{\alpha}, N \rangle = \omega(\Delta_M \bar{\alpha} - \Delta_{\partial M} \bar{\alpha}) - \bar{\alpha}(\delta(A(X^T))).$$

Now $\delta(A(X^T)) = (\delta A)(X^T) - \langle A, \delta^* X^T\rangle = -X^T(H) + \omega|A|^2$, where we have used the divergence constraint (3.3) and the fact that $(\delta^* X)^T = 0$. Also $-X^T(H) = -X(H) + \omega N(H) = -X(H) - \omega|A|^2 - \omega\lambda$. This gives

$$\delta(A(X^T)) = -X(H) - \lambda\omega,$$

and hence

$$\int_{\partial M} \langle \nabla_X d\bar{\alpha}, N\rangle = \int_{\partial M} -\bar{\alpha}\Delta_{\partial M}\omega + \bar{\alpha}X(H) + \bar{\alpha}\lambda\omega + \omega\Delta_M\bar{\alpha}.$$

Next, from (2.11), one computes $2A'_{\delta^* X} = \mathcal{L}_X A - D^2\omega$, so that

$$2trA'_{\delta^* X} = X(H) - \Delta\omega = 0,$$

which leaves

$$\int_{\partial M} \langle \nabla_X d\bar{\alpha}, N\rangle = \int_{\partial M} \omega(\Delta_M\bar{\alpha} + \lambda\bar{\alpha}).$$

Substituting this in (5.27) gives

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h\rangle = \int_{\partial M} \omega[-\tfrac{1}{2}\Delta_M\bar{\alpha} - \lambda\bar{\alpha} + \tfrac{1}{2}\chi] = 0,$$

where the last equality follows from (5.11).

Since (5.25) implies that $tr\tau'_{\delta^* X} = 0$, and since h is arbitrary at ∂M modulo pure-trace terms, it follows that

$$(\delta^* X)^T = 0, \quad \text{and} \quad A'_{\delta^* X} = 0,$$

at ∂M, and the result then follows as before from Proposition 2.4. ■

Proof of Corollary 1.2.

Theorem 1.1 implies that the isometry group $SO(n+1)$ of S^n extends to a group of isometries of the Einstein manifold (M^{n+1}, g). This reduces the Einstein equations to a simple system of ODE's, (the metric g is of cohomogeneity 1), and it is standard that the only smooth solutions are given by constant curvature metrics, cf. [4] for example. ■

The same proof shows that if $(\partial M, \gamma)$ is homogeneous, then any Einstein filling metric (M, g) is of cohomogeneity 1. Such metrics have been completely classfied in many situations, cf. [4] for further information.

We complete this section with a brief discussion of exterior and global boundary value problems. Thus, suppose M^{n+1} is an open manifold with compact "inner" boundary ∂M and with a finite number of ends, each (locally) asymptotically flat. Topologically, each end is of the form $(\mathbb{R}^k \setminus B) \times T^{n+1-k}$, or a quotient of this space by a finite group of isometries. Here T^{n+1-k} is the $(n+1-k)$-torus, and we assume $3 \leq k \leq n+1$. Assume also, as usual, that $\pi_1(M, \partial M) = 0$. An Einstein metric is asymptotically locally flat (ALF) if it decays to a flat metric on each end at a rate $r^{-(k-2)}$, (the decay rate of the Green's function for the Laplacian), where r is the distance from a fixed point.

It is proved in [3] that the analog of Theorem 2.1 holds, namely the space of asymptotically locally flat Einstein metrics on an exterior domain M is a smooth Banach manifold, for which the Dirichlet boundary map is C^∞ smooth. Lemma 2.3 also holds in this context. All of the remaining results in §2 - §5 above concern issues at or near ∂M, and it is straightforward to verify that their proofs carry over to this exterior context without change. In particular the analog of Theorem 1.1 holds:

Proposition 5.4. *Let g be a $C^{m,\alpha}$ Ricci-flat metric on an exterior domain M, $m \geq 3$, with a finite number of locally asymptotically flat ends. Suppose also* (1.2) *holds. Then any Killing field X on $(\partial M, \gamma)$ for which $X(H) = 0$, extends uniquely to a Killing field on (M, g).*

■

Next we point out that an analog of Theorem 1.1 holds for complete conformally compact Einstein metrics, where the boundary is at infinity, (conformal infinity). The proof below corrects an error in the proof of this result in [2].

Theorem 5.5. *Let (M, g) be a conformally compact Einstein metric, with smooth conformal infinity $(\partial M, [\gamma])$ and suppose $\pi_1(M, \partial M) = 0$. Then any (conformal) Killing field of ∂M extends to a Killing field of (M, g).*

Proof: The proof is a simple adaptation of the proof of Theorem 1.1, using information provided in [2], to which we refer for background details. Let t be a geodesic compactification of (M, g) and let $S(t)$ and $B(t)$ be the level and super-level sets of t, so that $\partial M = S(0)$, $M = B(0)$. The Killing vector field X on $(\partial M, \gamma)$ is extended into M to be in Bianchi gauge, so that $\delta^* X$ is transverse-traceless. One then has $\langle X, N\rangle = O(t^{n+1})$, where $N = -t\partial_t$ is the unit outward normal at $S(t)$, (cf. [2]). As in (5.21)-(5.22), one then has

$$\int_{S(t)} \langle \mathcal{L}_X \tau, h\rangle = \int_{S(t)} (Ric(N, X^T))'_h + O(t) = -2\int_{S(t)} \langle \nabla_{X^T} d\bar{\alpha}, N\rangle + O(t). \tag{5.29}$$

From [2, (5.43)],

$$\mathcal{L}_X A(e,e) = -\tfrac{n-2}{2} t^n \mathcal{L}_X g_{(n)}(\bar{e}, \bar{e}) + O(t^{n+1}),$$

where e is any unit vector with respect to g, while $\bar{e} = t^{-1}e$ is the corresponding unit vector for $\bar{g} = t^2 g$. The term $\mathcal{L}_X g_{(n)}(\bar{e}, \bar{e})$ is uniformly bounded as $t \to 0$. Taking the trace and using the fact that $tr g_{(n)}$ is intrinsically determined by γ, it follows that

$$X(H) = O(t^{n+1}).$$

Computing the right side of (5.29) as before following (5.22), one has to estimate the terms $\langle A, \delta^* X^T\rangle$ and $X(H)$. But $A = g|_{S(t)} + O(t^2)$ and the estimates above on X give $\langle A, \delta^* X^T\rangle = O(t^{n+1})$ and similarly for $X(H)$. Moreover, assuming the boundary values h_1, h_2 in (5.7) remain bounded as $t \to 0$, the term $\bar{\alpha}$ also remains bounded as $t \to 0$, by standard estimates applied to (5.19). Since the volume form of $S(t)$ is on the order of $O(t^{-n})$, it follows from (5.29) that

$$\int_{S(t)} \langle \mathcal{L}_X \tau, h\rangle \to 0 \ \text{ as } \ t \to 0.$$

It follows then that on ∂M,

$$\mathcal{L}_X g_{(n)} = 0,$$

so that the flow of X preserves both the boundary metric and $g_{(n)}$ term. The result then follows from the unique continuation result, Theorem 1.2, of [2].

■

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