1. Some answers to problems from §1.3

1a) Take the value of

$$|x| = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos{(2k+1)x}$$

at $x = 0$. Since $\cos 0 = 1$, we obtain

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\,,$$

and a simple algebraic manipulation of this result yields

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

2a) The function $f(x) = |x| + x$, $-1 < x < 1$, is the same as

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 2x & 0 \leq x < 1 \end{cases}\,,$$

function that is clearly continuous on $-1 < x < 1$. Its derivative exists everywhere on the interval except at $x = 0$, where it has a jump discontinuity. Explicitly, this derivative is given by

$$f'(x) = \begin{cases} 0 & -1 < x < 0 \\ 2 & 0 < x < 1 \end{cases}\,,$$

and so f' is sectionally continuous. Thus, f is sectionally smooth on $-1 < x < 1$. The theorem on convergence of the Fourier series applies, and we conclude that the Fourier series of f converges at any value of x and defines a periodic function of period 2. Moreover, this series converges to the average of the one sided limits $f(x^+)$ and $f(x^-)$ for $-1 < x < 1$. Since f is continuous on that interval, the Fourier series converges to $f(x)$ for $-1 < x < 1$. At $x = -1$, the Fourier series converges to the average of 2 and 0, that is to say, it converges to 1. Similarly, the Fourier series converges to 1 at $x = 1$.

3) Since f is continuous, we have $f(x^+) + f(x^-) = 2f(x)$ for any x. But f is also periodic and sectionally continuous, so the theorem on convergence of Fourier series applies. Thus, the Fourier series of f converges to the average

$$\frac{f(x^+) + f(x^-)}{2} = f(x)\,.$$

In other words, the Fourier series of such a function at x converges to $f(x)$, for any x.

2. Some answers to problems from §1.4

1b) The function $\sinh x = \frac{e^x - e^{-x}}{2}$ is continuous and has continuous derivatives everywhere on the interval $-\pi < x < \pi$. However, since $\sin{(-\pi)} \neq \sinh \pi$, its periodic extension is not continuous. The Fourier series of this function cannot converge uniformly in any interval containing $x = \pm\pi$.

2) The given function is *even* and has a removable singularity at $x = 0$, where it is not defined. We obtain a sectionally smooth even function by extending the definition of f to be 1 at $x = 0$ (1 is the limit of $\sin x/x$ as x goes to 0):

$$\tilde{f}(x) = \begin{cases} \dfrac{\sin x}{x} & -\pi < x < \pi\,, \quad x \neq 0\,, \\ 1 & x = 0\,. \end{cases}$$

The periodic extension of $\tilde{f}$ is continuous everywhere and sectionally smooth. Hence, its Fourier series converges to it uniformly.

4) Since the sequences a_n and b_n tend to zero as n tends to infinity, they must be bounded. Thus, there exists a constant C such that

$$|a_n| < M\,, \quad |b_n| < M\,, \quad \text{for all } n\,.$$

Therefore, the sequence of absolute values of the coefficients of the given series, $|e^{-\alpha n}a_n|$ and $|e^{-\alpha n}b_n|$, can both be compared from above to $Me^{-\alpha n}$:

$$\begin{aligned} |e^{-\alpha n}a_n| &< Me^{-\alpha n}\,, \\ |e^{-\alpha n}a_n| &< Me^{-\alpha n}\,. \end{aligned}$$

But the series $\sum_{n=1}^{\infty} Me^{-\alpha n}$ converges when α is a positive constant. Therefore

$$\sum_{n=1}^{\infty} |e^{-\alpha n}a_n| + |e^{-\alpha n}b_n| < 2\sum_{n=1}^{\infty} Me^{-\alpha n}\,,$$

and, by the comparison test, the series to the left converges as well. Hence, the given Fourier series converges uniformly on the whole real line.