

## HOMEWORK 2, MAT 568, FALL 06

Due: Thursday, Nov 9.

1. Let  $(M, g)$  be a Riemannian manifold and  $f : M \rightarrow M$  a diffeomorphism. Let  $\nabla^{f^*g}$  be the Levi-Civita connection for the metric  $f^*g$  and  $\nabla^g$  the Levi-Civita connection for the metric  $g$ . Prove that

$$\nabla^{f^*g} = f^*\nabla^g,$$

i.e.

$$\nabla_X^{f^*g} Y = \nabla_{f_*X}^g f_*Y.$$

From this, deduce that the (3, 1) Riemann curvature tensor transforms naturally under pullback:

$$R^{f^*g} = f^*R.$$

2. Let  $\tilde{g} = \lambda^2 g$ , where  $\lambda$  is a positive constant, so that  $\tilde{g}$  is a rescaling of  $g$ . Show that

$$\tilde{\nabla} = \nabla$$

and hence, if  $R$  denotes the (3, 1) Riemann curvature tensor, then

$$\tilde{R}(X, Y)Z = R(X, Y)Z.$$

On the other hand, if  $R$  is now the (4, 0) curvature tensor, then show

$$\tilde{R} = \lambda^{-2}R,$$

i.e.

$$\tilde{g}(\tilde{R}(X, Y)Z, W) = \lambda^{-2}g(R(X, Y)Z, W).$$

Deduce that the sectional, Ricci curvature on unit vectors, and scalar curvature transform under rescaling as  $\lambda^{-2}$ .

(\*) Extra Credit: Deduce the formula for the transformation of the curvature under conformal changes.

3. Now let  $\tilde{g} = u^2 g$  be a conformal change of the metric  $g$ . Prove that

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\log u)Y + Y(\log u)X - g(X, Y)\nabla \log u.$$

4. Let  $G$  be a Lie group with a bi-invariant Riemannian metric, i.e. the metric is invariant under right and left translations of the group. It follows that inner automorphisms  $i_h(g) = h^{-1}gh$  are isometries of the metric and hence the adjoint action of the Lie algebra  $\mathcal{L}(G)$  is skew-symmetric, i.e.

$$ad_U : \mathcal{L}(G) \rightarrow \mathcal{L}(G), \quad ad_U(X) = [X, U]$$

satisfies

$$\langle [X, U], Y \rangle = -\langle X, [Y, U] \rangle.$$

Use this to prove that

(a).  $\nabla_X Y = \frac{1}{2}[X, Y]$ .

(b).  $R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]$ .

(c).  $\langle R(X, Y)Z, W \rangle = -\frac{1}{4}\langle [X, Y], [Z, W] \rangle$ .

Conclude that all the sectional curvatures are non-negative. Show that  $Ric(X) = 0$  if and only if  $X$  commutes with all other left-invariant vector fields, i.e.  $X$  is in the center of  $\mathcal{L}(G)$ .

5. Let  $M^{n-1}$  be a hypersurface in  $\mathbb{R}^n$  with the induced metric, and suppose a local patch of  $M$  is given as the graph of a function  $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ . Suppose that  $f(0) = 0$ , so that the origin  $0 \in M$  and  $Df(0) = 0$ , so that the tangent space to  $M$  at  $0$  is  $\mathbb{R}^{n-1}$ .

Show that the 2nd fundamental form of  $M$  at  $0$  is proportional to the Hessian of  $f$ :

$$A = \frac{1}{|\nabla f|} D^2 f.$$

6. Consider the hypersurface  $M$  in  $\mathbb{R}^{n+1}$  given by

$$x^{n+1} = (x^n)^2,$$

with the induced Riemannian metric.

Prove that  $M$  is isometric to  $\mathbb{R}^n$  with the flat metric, i.e. it is flat.

On the other hand, prove that the 2nd fundamental form  $A$  of  $M$  does not vanish anywhere.