HOMEWORK I, MAT 568, FALL 06

Due: Tuesday, Oct 24.

Reminder: No class the week of Oct 9-13. This will be made up at semester end, as needed. Classes for week of Oct 16-20, usual place and time, given by Satyaki Dutta and Andrew Bulawa.

If you've not already done so, read and understand all of Chapter 1 of the Petersen text. Read also the proof of the Hopf-Rinow theorem (Section 5.7 of Petersen). In fact, it would be a good idea to read most of Section 5 of the text, even if it uses things we've not yet fully developed in class.

The following 3 problems from Section 5 of Petersen text.

1. Show that any homogeneous manifold (M, g), (i.e. the isometry group acts transitively on M), is necessarily geodesically complete.

2. Show that any smooth manifold admits a complete Riemannian metric.

3. Show that in any Riemannian manifold (M, g), one has

$$d(exp_p(tv), exp_p(tw)) = |t||v - w| + O(t^2),$$

where d is the distance function and $|v|$ is the g-norm of v.

4. A Killing field is a vector field X such that $\mathcal{L}_X g = 0$, where $\mathcal{L}$ is the Lie derivative. Show that X is Killing if and only if the associated 1-parameter group of diffeomorphisms consists of isometries.

The following (technical) problem is needed for Problem 6.

5. Let $\gamma : [a, b] \to M$ be a geodesic in (M, g) and let $p : [\alpha, \beta] \to [a, b]$ be a diffeomorphism, so that $c = \gamma \circ p$ is a reparametrization of γ. Show that c satisfies

$$\frac{d^2 c^k}{dt^2} + \sum_{ij} \Gamma_{ij}^k(c(t)) \frac{dc^i}{dt}\frac{dc^j}{dt} = \frac{dc^k}{dt}\frac{p''(t)}{p'(t)}.$$

Conversely, show that if c satisfies this equation, then γ is a geodesic.

6. The Poincaré half-plane is the manifold $(\mathbb{R}^2)^+ = \{(x, y) : y > 0\}$, with Riemannian metric

$$g = \frac{1}{y^2}(dx^2 + dy^2).$$

(a). Compute that

$$\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{1}{y},\ \Gamma_{11}^1 = \frac{1}{y}, \text{ and all other} \Gamma = 0.$$

(b). Let $c(t) = (t, y(t))$ be a semicircle in the half-plane with center at $(0, y_0)$ of radius R. Show that

$$\frac{d^2 y}{dt^2} = -\frac{y'(t)}{t - y_0} - \frac{y'(t)^2}{y(t)}.$$

(c). Using Problem 5, show that all geodesics in the Poincaré half-plane are reparametrizations of semi-circles with center on the x-axis, together with straight lines parallel to the y-axis.

(d). Show that these geodesics have infinite length in either direction, so that the upper half plane is complete in this metric. Is this true for the Euclidean metric?

(e). Finally, show that the linear fractional transformations

$$f(z) = \frac{az+b}{cz+d}$$

mapping the upper half plane to itself, are isometries of the Poincaré metric. Conclude that this metric is homogeneous.