MAT 211, Linear Algebra, Summer 1, 2008

Homework 5 - due Monday, July 7

Important Note: **Please show all your work and provide full justification whenever the case arises. Please write clearly. Answers with little or no justification (when the case arises) will receive no credit. Similarly, if you do not show all your work at some problem, your answer will receive no credit. Answers that are unclearly written might not be graded.** Problems that are accompanied by an exclamation mark (!) are considered to be more challenging (or nonstandard). These problems are also mandatory. In the exams problems with higher level of difficulty or nonstandard will also be marked with an exclamation mark. Please note that in some cases, a marked nonstandard problem might be easier and shorter than the standard ones. The problems that are not explicitly written are from the official textbook.

1. (!) If A is an $(n \times n)$ matrix with complex entries and $f(X) = a_0 + a_1 X + a_2 X^2 + ... + a_k X^k$ is a polynomial with complex coefficients, we say that A is a root of f if

$$a_0 I_n + a_1 A + a_2 A^2 + ... + a_k A^k = O,$$

where O is the zero matrix and I_n is the identity matrix.
In order to familiarize yourself with this new concept, please consider the following examples:
(1.1) Matrix $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ is a root of the polynomial X^3 because

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}^3 = O.$$

(1.2) The zero matrix O is, trivially, the root of the polynomial X.
(1.3) Matrix $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ is a root of the polynomial $X^2 - 10X - 2$ because

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}^2 - 10 \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(1.4) I_n is a root of the polynomial $X^2 - 1$ because

$$I_n^2 - I_n = O.$$

(1.5) Obviously I_n is a root of $X - 1$ as well.
Remark. So a matrix may be a root of several polynomials. But now a

natural question appears: is any square matrix the root of some non-zero polynomial?
Recall the following definition. If A is an $(n \times n)$ matrix with complex entries, the characteristic polynomial of A is, by definition, $P_{\text{A}}(X) = det(XI_n - A)$ (You proved that $P_A(X)$ is a polynomial of degree n in homework 2.)
The following theorem holds.
The Hamilton-Cayley Theorem. Any square matrix with complex entries is the root of its own characteristic polynomial.
(a) Prove the Hamilton-Cayley theorem for the case when the matrix has 2 rows and 2 columns.

(b) Now assume the Hamilton-Cayley theorem in the general case, without proving it. Prove that for all $(n \times n)$ matrices with complex entries A, $A^k \in Span\{I_n, A, A^2, ..., A^{n-1}\} \forall k \geq 0$.
Hint: What do we usually use when we want to prove a claim that depends on a non-negative integer?

2. Let V be a real linear space.
(a) Let $\{V_{\text{i}}\}_{\text{i}\in I}$ be a family of linear subspaces of V. Prove that

$$\bigcap_{i\in I} V_i$$

is a linear subspace of V.
(b) Let S be a non-empty subset of V. We define the span of S (denoted $Span(S)$) to be the following subset of V:

$$Span(S) = \bigcap_{Z \subseteq V, Z\ linear\ subspace\ of\ V, Z \supseteq S} Z.$$

First, please note that $Span(S)$ makes sense: it is the intersection of all linear subspaces of V that contain S (and such linear subspaces do exist - for instance, V is a linear subspace of V that contains S). Now prove that $Span(S)$ is a linear subspace of V that contains S.
(c) At this point, the purpose will be to prove that "$Span(S)$ is the smallest (in the sense of inclusion of linear subspaces) linear subspace of V that contains set S." Prove that if Z is a linear subspace of V, the following equivalence holds:

$$Z \supseteq S \Longleftrightarrow Z \supseteq Span(S).$$

Hints: Whenever you want to prove an equivalence, you have to prove two implications. Thus, first prove that "$\Longrightarrow$" holds; in other words, first assume that $Z \supseteq S$ and prove that, under this assumption, $Z \supseteq SpanS$. Then prove the converse implication, namely "$\Longleftarrow$" - that is, this time,

assume that $Z \supseteq Span(S)$ and prove that, under this assumption, $Z \supseteq S$.
(d) Prove that the following equality of sets holds:

$$Span(S) = \{\sum_{i=1}^{k} x_i s_i : x_1, x_2, ..., x_k \in \mathbb{R}, s_1, s_2, ..., s_k \in S, k \geq 1\}.$$

Hint: Whenever you want to prove that two sets are equal, you do this by "double inclusion". For instance, in this case, first prove that

$$Span(S) \supseteq \{\sum_{i=1}^{k} x_i s_i : x_1, x_2, ..., x_k \in \mathbb{R}, s_1, s_2, ..., s_k \in S, k \geq 1\}.$$

Then prove the reversed inclusion, namely prove that

$$Span(S) \subseteq \{\sum_{i=1}^{k} x_i s_i : x_1, x_2, ..., x_k \in \mathbb{R}, s_1, s_2, ..., s_k \in S, k \geq 1\}.$$

In order to prove the $\subseteq$inclusion first note that

$$S \subseteq \{\sum_{i=1}^{k} x_i s_i : x_1, x_2, ..., x_k \in \mathbb{R}, s_1, s_2, ..., s_k \in S, k \geq 1\}.$$

Then prove that

$$\{\sum_{i=1}^{k} x_i s_i : x_1, x_2, ..., x_k \in \mathbb{R}, s_1, s_2, ..., s_k \in S, k \geq 1\}$$

is a linear subspce of V (so it is a linear subspace of V that contains S). Now you can finish proving this inclusion by using (c). For the other inclusion, ($\supseteq$) simply take an arbitrary element of

$$\{\sum_{i=1}^{k} x_i s_i : x_1, x_2, ..., x_k \in \mathbb{R}, s_1, s_2, ..., s_k \in S, k \geq 1\},$$

that is, pick arbitrary $k \geq 1, x_1, x_2, ..., x_k \in \mathbb{R}$,$s_1, s_2, ..., s_k$$\in S$ and prove that

$$\sum_{\mathrm{i}=1}^{k} x_i s_i \in Span(S).$$

3. (!) Find

$$dim_{\mathbb{R}} Span\{AB - BA : A, B \in M_{\mathrm{n}}(\mathbb{R})\}.$$

Hint: Use the canonical basis of $M_n(\mathbb{R})$. If the phenomenon is not clear, *analyse some particular cases of n(this is a general technique, when we wish to handle a "general" complicated problem, it is often useful to study particular cases).*

4. Let (A_1, A_2, A_3, A_4) and (B_1, B_2, B_3, B_4) be two elements of $\mathbb{R}^4$. Consider the following subset of $\mathbb{R}^4$:

   $$Sol = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : A_1x_1+A_2x_2+A_3x_3+A_4x_4 = 0, B_1x_1+B_2x_2+B_3x_3+B_4x_4 = 0\}.$$

   (a) Prove that Sol is a linear subspace of $\mathbb{R}^4$.

   (!) (b) Find the dimension of Sol in terms of the rank of $\begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \end{pmatrix}$.

   Prove all your claims. You may NOT use the general theorem stated in class.

   Hint: In your solution for part (b), please analyse the 3 possible cases corresponding to the 3 possible values of the rank of $\begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \end{pmatrix}$.

5. Compute the kernel of the linear transformations given in exercises 6 and 22 on page 169, section 4.2. For each transformation, find a basis of the kernel, compute the dimension of the kernel, compute the dimension of the domain of the linear transformation and then apply the rank-nullity theorem to compute the dimension of the image of the linear transformation. Decide whether the transformations are: one-to-one, onto, isomorphisms. Prove all your claims.

6. Solve problems 12, 20, 42, 46 and 48 on pages 314 - 316, section 7.2.

7. Solve problems 8, 18, 20, 28 and 38 on pages 324 - 326, section 7.3.

8. Solve problems 24 and 26 on page 338, section 7.4.

9. Solve problems 26 and 28 on page 351, section 7.5.

10. (!) Let V be an $F-$vector space, where $F \in \{\mathbb{R}, \mathbb{C}\}$. Let

    $$\{v_1, v_2, ..., v_n\} \subseteq V.$$

    (a) Assume that $n \geq 2$. Prove that the following are equivalent:

    (A) $v_1, v_2, ..., v_n$ are linearly independent.

    (B) $\forall$i$\in \{1, 2, ..., n\}$,$v_i \notin Span(\{v_j : j \in \{1, 2, ..., n\}, j \neq i\})$. (In other words, no vector in $\{v_1, v_2, ..., v_n\}$can be written as a linear combination of the other vectors in $\{v_1, v_2, ..., v_n\}$.)

(C) $v_1 \neq 0_V$ and $\forall i \geq 2, v_i \notin Span(\{v_j : 1 \leq j \leq i-1\})$.

(b) Assume that $n = 1$. Prove that

$$v_1 \, is \, linearly \, independent \Leftrightarrow v_1 \neq 0_V.$$

Hints: For part (a), you must prove (A)$\Leftrightarrow$ (B) and (A)$\Leftrightarrow$ (C). (A) $\Leftrightarrow$ (B) is very easy. For (A) $\Leftrightarrow$ (C), one of the implications is almost obvious; for the non-trivial implication, prove by induction on k that $\{v_1, v_2, ..., v_k\}$ are linearly independent $\forall k, 1 \leq k \leq n$.