MAT 211, Linear Algebra, Summer 1, 2008

Homework 4 - due Wednesday, July 2

Important Note: **Please show all your work and provide full justification whenever the case arises. Please write clearly. Answers with little or no justification (when the case arises) will receive no credit. Similarly, if you do not show all your work at some problem, your answer will receive no credit. Answers that are unclearly written might not be graded.** Problems that are accompanied by an exclamation mark (!) are considered to be more challenging (or nonstandard). These problems are also mandatory. In the exams problems with higher level of difficulty or nonstandard will also be marked with an exclamation mark. Please note that in some cases, a marked nonstandard problem might be easier and shorter than the standard ones. The problems that are not explicitly written are from the official textbook.

1. Solve problems 4, 6, 8 and 10 on page 199, section 5.1.
2. Solve problem 18 on page 200, section 5.1.
3. Solve problems 26, 28, 30 and 32 on page 201, section 5.1.
4. Solve problems 40, 42, 44 and 46 on page 201, section 5.1.
5. Solve problems 2,4,6,8,10 and 12 on page 208, section 5.2. In addition, for each of these find the change of basis matrix from the initial basis to the orthonormal one.
6. Solve problems 2,4, 32, 34 and 36 on pages 216 - 217, section 5.3
7. Solve problem 52 on page 218, section 5.3. A matrix is called symmetric if it is square and equal to its transpose.
8. Solve problem 56 on page 218, section 5.3.
9. Let S^{n} denote the subset of $\mathbb{R}^{\mathrm{n}+1}$defined by

$$S^{\mathrm{n}} = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{\mathrm{n}+1} : \sum_{i=1}^{n+1} x_{\mathrm{i}}^2 = 1\}.$$

   Let $y = (y_1, y_2, ..., y_{\mathrm{n}+1}) \in S^{\mathrm{n}}$. We let $T_{\mathrm{y}}S^{\mathrm{n}}$denote the following subset of $\mathbb{R}^{\mathrm{n}+1}$

$$T_{\mathrm{y}}S^{\mathrm{n}} = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{\mathrm{n}+1} : \sum_{\mathrm{i}=1}^{\mathrm{n}+1} x_{\mathrm{i}} y_{\mathrm{i}} = 0\}.$$

   Prove that $T_{\mathrm{y}}S^{\mathrm{n}}$is a linear subspace of $\mathbb{R}^{\mathrm{n}+1}$ and compute its dimension. *Note: For persons interested in Math, S^n is the $n-$dimensional sphere and T_yS^n is the tangent space to S^n at point y. This is not the definition of the tangent space to the sphere, but a consequence of a deep theorem.*

10. (!) *(from Ornea and Turtoi, "An Introduction to Geometry", Theta, 2000)* Let $(E, <, >)$ be a (real) inner product space of finite dimension. Let $f : E \to E$ be a map such that

$$< f(x), f(y) >=< x, y > \forall x, y \in E.$$

Show that f is a linear transformation.

11. (!) *(from L.Ornea and A.Turtoi, "An Introduction to Geometry", Theta, 2000)*
Let V be a real linear space of finite dimension. Let $J : V \to V$ be a linear transformation such that

$$J \circ J = -identity.$$

Prove that $dim_{\mathbb{R}} V$ is even.

12. (!) Read Fact A.10 on page 440 in the textbook (if you know the basics about cross product in $\mathbb{R}^3$you do not need to read that).
(from L.Ornea and A.Turtoi, "An Introduction to Geometry", Theta, 2000)
Let x, y, z be vectors in $\mathbb{R}^3$.
If $x \times y + y \times z + z \times x = 0$ then $\{x, y, z\}$are linearly dependent.

13. (!) If $(E, <, >)$is an inner product space, for any $x \in E$,we define

$$\| x \|= \sqrt{< x, x >}.$$

(from L.Ornea and A.Turtoi, "An Introduction to Geometry", Theta, 2000)
(The Gram Determinant) Let $(E, <, >)$ be an $n-$dimensional inner product space and $x_1, x_2, ..., x_k \in E$. The determinant

$$G(x_1, x_2, ..., x_k) = \begin{vmatrix} < x_1, x_1 > & < x_1, x_2 > & ... & < x_1, x_k > \\ < x_2, x_1 > & < x_2, x_2 > & ... & < x_2, x_k > \\ ... & ... & ... & ... \\ < x_k, x_1 > & < x_k, x_2 > & ... & < x_k, x_k > \end{vmatrix}$$

is called the Gram determinant of the given vectors $x_1, x_2, ..., x_k \in E$. Prove that

$$0 \leq G(x_1, x_2, ..., x_k) \leq \| x_1 \|^2 \| x_2 \|^2 ... \| x_k \|^2 .$$

Prove that $0 = G(x_1, x_2, ..., x_k)$holds if and only if $x_1, x_2, ..., x_k$ are linearly dependent. Prove that

$$G(x_1, x_2, ..., x_k) =\| x_1 \|^2 \| x_2 \|^2 ... \| x_k \|^2$$

holds if and only if $< x_i, x_j >= 0 \forall i \neq j$.
My hint: What do we usually use when we want to prove a claim that depends on a non-negative integer?

14. (!) *(from L.Ornea and A.Turtoi, "An Introduction to Geometry", Theta, 2000)*
(The volume of the k-dimensional parallelepiped)
Let $x_1, x_2, ..., x_k$ be fixed elements (vectors) in the $n-$dimensional inner product space $(E, <, >)$. Assume $k \geq 2$. Let

$$U_j = Span(\{x_1, x_2, ..., x_j\}).$$

Let x'_{j+1}denote the orthogonal projection of x_{j+1}onto U_j and

$$h_j = \left\| x_{j+1} - x'_{j+1} \right\| \forall j \in \{1, 2, ..., k-1\}.$$

(h_j is the distance from x_{j+1} to U_j). Let

$$V(x_1, x_2, ..., x_k) = \|x_1\| \, h_1 h_2 ... h_{k-1}.$$

Prove that

$$V(x_1, x_2, ..., x_k)^2 = G(x_1, x_2, ..., x_k).$$

Note: $G(x_1, x_2, ..., x_k)$ was defined in the previous problem.
Let $f : E \to E$ be a linear transformation. Assume that $k = n$ and $x_1, x_2, ..., x_n$ are linearly independent. Prove that

$$k(f) = \frac{V(f(x_1), f(x_2), ..., f(x_n))}{V(x_1, x_2, ..., x_n)}$$

is constant; in other words, prove that $k(f)$ does not depend on the chosen basis $\{x_1, x_2, ..., x_n\}$ but only on the linear transformation f.
My hint: Compute ^tAGA, *where* A *is the matrix of* f *with respect to basis* $\{x_1, x_2, ..., x_n\}$ *and* G *is the matrix that appears in the Gram determinant associated to* $\{x_1, x_2, ..., x_n\}$.