MAT 211, Linear Algebra, Summer 1, 2008

Homework 3 - due Thursday, June 26

Important Note: **Please show all your work and provide full justification whenever the case arises. Please write clearly. Answers with little or no justification (when the case arises) will receive no credit. Similarly, if you do not show all your work at some problem, your answer will receive no credit. Answers that are unclearly written might not be graded.** Problems that are accompanied by an exclamation mark (!) are considered to be more challenging (or nonstandard). These problems are also mandatory. In the exams problems with higher level of difficulty or nonstandard will also be marked with an exclamation mark. Please note that in some cases, a marked nonstandard problem might be easier and shorter than the standard ones. The problems that are not explicitly written are from the official textbook.

1. Prove that $\{(1,0,1),(2,3,5),(-1,1,1)\}$ is a basis of $\mathbb{R}^3$ and write $(5,2,6)$ as a linear combination of the vectors of this basis.

2. For k a non-negative integer, we define P_{k} to be the space of all polynomials of degree less than or equal to k with real coefficients. For instance, if $k = 2$,
$$P_2 = \left\{a + bX + cX^2 : a, b, c \in \mathbb{R}\right\}.$$
Thus, $1 + 2X \in P_2$, $1 - 10X^2 \in P_2$, but $1 - X^2 + 50X^3 \notin P_2$.
Do exercises 2,8,10, 16, 18, 20, 22 and 34 in section 4.1, page 162. For exercise 22 one needs the following definition. A diagonal $(n \times n)$ matrix (with real entries) is a matrix of the form
$$\begin{pmatrix} x_1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & x_2 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & x_3 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & x_{n-2} & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & x_{n-1} & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & x_n \end{pmatrix}, x_i \in \mathbb{R}, \forall i \in \{1, 2, 3, ..., n-2, n-1, n\}.$$
For instance, if $n = 3$, the space of all diagonal (3×3) matrices with real entries is
$$\left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

3. Do exercise 58 in section 4.1, page 163.

4. Do exercises 2, 6,12, 18, 26 and 32 in section 4.2, page 169.

5. Do exercises 34, 36, 42, 48 in section 4.2, page 170.

6. Do exercise 76 in section 4.2, page 171. Please note that (though easy) problem 76 states a fundamental property of linear transformations. You are expected to know it and to be able to prove it at any time.

7. Let A be an $(n \times n)$ matrix with real entries, $A = (a_{\mathrm{i,j}})_{1 \le i \le n, 1 \le j \le n}$. We define the trace of A, $TrA \in \mathbb{R}$, by

$$TrA = \sum_{\mathrm{i=1}}^{\mathrm{n}} a_{\mathrm{ii}}.$$

For instance, if $n = 3$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

then

$$TrA = a_{11} + a_{22} + a_{33} \in \mathbb{R}.$$

(a) Prove that

$$Tr(AB) = Tr(BA) \forall A, B \in M_{\mathrm{n}}(\mathbb{R}).$$

(b) Let A and B be two $(n \times n)$ matrices with real entries such that $Tr(AB) \neq 0$. Let $\lambda \in \mathbb{R}$ be such that

$$AB = \lambda BA.$$

Prove that $\lambda = 1$.

(c) Let A and B be two $(n \times n)$ invertible matrices with real entries and let $\lambda \in \mathbb{R}$ be such that

$$AB = \lambda BA.$$

Prove that $\lambda \in \{1, -1\}$.

8. (!) Let V be an $\mathbb{R}$- vector space (or a real linear space) and $\{v_1, v_2, v_3, ..., v_{\mathrm{n-2}}, v_{\mathrm{n-1}}, v_n\}$ a basis of V, $n \geq 3$. Prove that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, ..., v_{\mathrm{n-2}} + v_{\mathrm{n-1}}, v_{\mathrm{n-1}} + v_{\mathrm{n}}, v_{\mathrm{n}} + v_1\}$ is a basis of V if and only if n is odd.

9. In class we have defined matrices with real entries. Similarly, one can define matrices with complex entries. More precisely, an $(n \times m)$ matrix with complex entries is a function

$$A = (a_{ij})_{1 \le i \le n, 1 \le j \le m} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{C}.$$

Here $a_{\mathrm{ij}} =^{by\, definition} A(i, j) \forall\ (i, j) \in \{1, 2, ..., n\} \times \{1, 2, ..., m\}$.
As in the real case, we represent matrix A by an array of complex numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,(m-1)} & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,(m-1)} & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3,(m-1)} & a_{3m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{(n-1),1} & a_{(n-1),2} & a_{(n-1),3} & \dots & a_{(n-1),(m-1)} & a_{(n-1),m} \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{n,(m-1)} & a_{nm} \end{pmatrix}.$$

As for matrices with real entries, we define the product of matrices with complex entries as follows. If $A = (a_{ij})_{1 \le i \le n, 1 \le j \le m}$ is an $(n \times m)$ matrix with complex entries and $B = (b_{jk})_{1 \le j \le m, 1 \le k \le r}$ is an $(m \times r)$ matrix with complex entries, we define the product between A and B, denoted by $AB = (c_{ik})_{1 \le i \le n, 1 \le k \le r}$ to be the $(n \times r)$ matrix with complex entries whose entries are given by

$$c_{\mathrm{ik}} = \sum_{\mathrm{s}=1}^{\mathrm{m}} a_{\mathrm{is}} b_{\mathrm{sk}} \forall i \in \{1, 2, ..., n\}, \forall k \in \{1, 2, ..., r\}.$$

The set of all $(n \times m)$ matrices with complex entries will be denoted by $M_{(n \times m)}(\mathbb{C})$ or $M_{n,m}(\mathbb{C})$. The set of all $(n \times n)$matrices with complex entries will be denoted by $M_n(\mathbb{C})$; these matrices are called square matrices (or, more precisely, complex square matrices). Let $A \in M_n(\mathbb{C})$. A is called invertible (or non-singular) if and only if $\exists B \in M_n(\mathbb{C})$such that $AB = BA = I_n$. If $A \in M_n(\mathbb{C})$ is invertible, then its inverse is unique; more precisely, $\exists!$matrix $A^{-1} \in M_n(\mathbb{C})$such that $AA^{-1} = A^{-1}A = I_n$. Moreover, the algorithm of computing the inverse of a non-singular matrix with complex entries is exactly the same as for matrices with real entries.

Let

$$\mathbb{H} = \left\{ \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

(9.1) Prove that $\mathbb{H}$ is an $\mathbb{R}$- linear subspace of $M_2(\mathbb{C})$. (You do not need to prove in the homework that $M_2(\mathbb{C})$ is a real vector space, but you should do this for yourself).
(9.2) Let

$$1^{\rightarrow} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; i^{\rightarrow} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

(Usually, $1^{\rightarrow}$ is simply denoted by 1 and $i^{\rightarrow}$ is simply denoted by i. We have chosen this notation to emphasize that, while 1 and i are complex numbers, $1^{\rightarrow}$and $i^{\rightarrow}$ are (2×2) matrices with complex entries. Also, with our previous notation, $1^{\rightarrow}= I_2$.)
Prove that $\{1^{\rightarrow}, i^{\rightarrow}, j, k\}$ is a basis for $\mathbb{H}$ as a real vector space.
(9.3) Check that

$$(i^{\rightarrow})j = -j(i^{\rightarrow}) = k, jk = -kj = (i^{\rightarrow}), k(i^{\rightarrow}) = -(i^{\rightarrow})k = j, (i^{\rightarrow})^2 = j^2 = k^2 = -1^{\rightarrow}.$$

(9.4) Prove that any non-zero matrix $A \in \mathbb{H}$ is invertible as a (2×2) matrix with complex entries and compute its inverse.
Note: The elements of $\mathbb{H}$ are called quaternions. $\mathbb{H}$ is named so to honor the memory of Sir William Rowan Hamilton who discovered the quaternions. For persons interested in Math, $\mathbb{H}$ is the classical example of a

non-commutative field. (The word "field" in this sentence is related to the property of the quaternions proved in (9.4).)

10. Compute the determinant of the following $(n \times n)$matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & ... & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & ... & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & ... & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & ... & 1 & 1 & 1 \\ ... & ... & .... & ... & ... & ... & ... & ... \\ 1 & 1 & 1 & 1 & ... & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & ... & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & ... & 1 & 1 & 0 \end{pmatrix}.$$

11. (!) *This problem is very important as it covers most of the concepts and results you have studied so far and because it provides a proof for the equivalence between 3 definitions for the rank of a matrix. Though difficult, if you do it, you will get 40% of the total score for this homework.*

Let V be a real vector space. The following fundamental theorems hold:
(A) If $dim_{\mathbb{R}}V = n$ and $\{v_1, v_2, ..., v_n\}$ is a set of n linearly independent elements of V, then $\{v_1, v_2, ..., v_n\}$ is a basis of V.
(B) If $Span\{e_1, e_2, ..., e_n\} = V$ then V is of finite dimension and $n \geq dimV$.
(c) If$\{f_1, f_2, ..., f_r\}$ is a system of r linearly independent elements of V and $Span\{e_1, e_2, ..., e_n\} = V$ then $r \leq dimV \leq n$. If $r < dimV$, there exists $\{i_1, i_2, ..., i_{dimV-r}\} \subseteq \{1, 2, ..., n\}$such that $\{f_1, f_2, ..., f_r\} \bigcup \{e_{i_1}, e_{i_2}, ..., e_{i_{dimV-r}}\}$ is a basis of V.

Recall the following definition from the lecture. Let $A = (a_{ij})_{i,j} \in M_n(\mathbb{R})$. A $k-minor$ of A is any determinant of any $(k \times k)$ matrix obtained from A by intersecting k rows and k columns. For instance, if

$$A = \begin{pmatrix} 1 & 2 & 4 & 0 \\ -1 & 2 & 5 & 6 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{pmatrix},$$

a 2- minor of A (obtained by intersecting rows 1 and 2 with columns 2 and 3 respectively) is

$$\begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix} = 2.$$

Another $2-$minor of A (obtained by intersecting rows 2 and 3 with columns 1 and 4 respectively) is

$$\begin{vmatrix} -1 & 6 \\ 0 & 1 \end{vmatrix} = -1.$$

Prove the following:
(11.1) Let $A = (a_{ij})_{i,j} \in M_n(\mathbb{R})$. Prove that the rows of A, viewed as vectors of $\mathbb{R}^{\mathrm{n}}$ are linearly independent if and only if $detA \neq 0$.
For instance, if

$$n = 3, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

one has to prove that $\{(a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}), (a_{31}, a_{32}, a_{33})\} \subseteq \mathbb{R}^3$ is a linearly independent system of vectors if and only if $detA \neq 0$.
(11.2) Formulate and prove a similar statement as in (11.1) for columns.
(11.3) Let $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq m} \in M_{\mathrm{n}\times m}(\mathbb{R})$. Assume that $n \leq m$. Prove that the rows of A viewed as vectors in $\mathbb{R}^{\mathrm{m}}$ are linearly independent if and only if A has a non-zero $n-$minor. Formulate a similar statement for case $n \geq m$ and similar statements for columns. Clearly the same type of arguments will show that your statements also hold.
(11.4) Let $A\in M_{n\times m}(\mathbb{R})$. Let C be the span of the columns of A and R be the span of the rows of A. (This means, for instance, that C is a linear subspace of $\mathbb{R}^{\mathrm{n}}$ whose elements are linear combinations of the columns of A.) Let $rankA$ denote, as in class:

$$rankA = max\{k \geq 0 : \exists\Delta\, a\, non - zero\, k - minor\, of\, A\}.$$

Prove that $rankA = dim_{\mathbb{R}}C = dim_{\mathbb{R}}R$.

EXTRA CREDIT (5%) *Please read the extra credit policy on the course website.*

The definition of a complex vector space (or complex linear space, or $\mathbb{C}$-vector space) is similar to the definition of a real vector space (or real linear space). All the fundamental linear algebra concepts that have been introduced so far over $\mathbb{R}$ - basis, linear transformation, the matrix and determinant associated to a linear transformation - have similar definitions in the complex case; one only has to replace $\mathbb{R}$ by $\mathbb{C}$ and "real" by "complex". For instance, if V and W are complex vector spaces, a complex linear transformation $T : V \to W$ is a map that satisfies 2 properties:
(a) $T(v + z) = T(v) + T(z)\forall v, z \in V$ and
(b) $T(\lambda v) = \lambda T(v)\forall\lambda \in \mathbb{C}, v \in V$.

Please note that any complex vector space can be naturally viewed as a real vector space (since any real number is a complex number as well).
(1) Let V be a complex vector space. Assume that $\{v_1, v_2, ..., v_n\}$ is a basis of V as a complex vector space. Prove that $\{v_1, iv_1, v_2, iv_2, ..., v_n, iv_n\}$is a basis of V viewed as a real vector space.
(2) Let W be a real vector space. We define

$$W^{\mathbb{C}} = \{(v, w) \in W \times W\}.$$

On $W^{\mathbb{C}}$ we define a natural addition

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \forall (v_1, w_1), (v_2, w_2) \in W^{\mathbb{C}}.$$

In addition to that, we define a scalar multiplication with complex numbers (scalars) as follows:

$$(a + ib)(v, w) = (av - bw, aw + bv) \forall a, b \in \mathbb{R}, \forall (v, w) \in W^{\mathbb{C}}.$$

Prove that together with this addition and complex scalar multiplication, $W^{\mathbb{C}}$ is a complex vector space.
(3) Let W be a real vector space and $W^{\mathbb{C}}$ be the complex vector space defined above; now view $W^{\mathbb{C}}$ as a real vector space. Prove that the map $T : W \to W^{\mathbb{C}}$ defined by $T(v) = (v, 0) \forall v \in W$ is a one-to-one real linear transformation.

(4) Let V be a complex vector space of finite complex dimension and $R : V \to V$ be the complex linear transformation defined by

$$R(v) = iv \forall v \in V.$$

Now view R as a real linear map and compute $det R$, where R is viewed as a real linear map between real linear spaces.