MAT 211, Linear Algebra, Summer 1, 2008

Homework 2 - due Monday, June 16

***Important Note:* Please show all your work and provide full justification whenever the case arises. Please write clearly. Answers with little or no justification (when the case arises) will receive no credit. Similarly, if you do not show all your work at some problem, your answer will receive no credit. Answers that are unclearly written might not be graded.** Problems that are accompanied by an exclamation mark (!) are considered to be more challenging (or nonstandard). These problems are also mandatory. In the exams problems with higher level of difficulty or nonstandard will also be marked with an exclamation mark. Please note that in some cases, a marked nonstandard problem might be easier and shorter than the standard ones. The problems that are not explicitly written are from the official textbook.

1. Do problems 2,4,34 and 46 in section 1.3, pages 35, 37.

2. Do problems 22 and 24 in section 6.3, page 289. Also, find the inverses of the matrices in exercises 30 and 32, section 6.3, page 289.

3. Do problems 36 and 38 in section 6.3, page 289.

4. (i) Do problem 38, section 6.2, page 272.
   (ii) Prove that $det(A^tA) \geq 0 \forall A$ an $(n \times n)$ matrix with real entries. Prove that equality holds $\Longleftrightarrow det A = 0$.

5. Read the definition of the dot product in $\mathbb{R}^n$on page 22, section 1.2, problem 34. Also, read the definition of perpendicularity in the same problem. If $v \in \mathbb{R}^n$, we define its norm by $\|v\|=\sqrt{< v, v >}$.
   (i) Let $v \in \mathbb{R}^n$. Prove the following claims:
   (a)
   $$\|v\| = 0 \Longleftrightarrow v = 0.$$
   (b) $\| \lambda v \|=| \lambda || v \| \ \forall \lambda \in \mathbb{R}, \forall v \in \mathbb{R}^n$.
   (ii) An $(n \times n)$ matrix A with real entries is called orthogonal if and only if any two different rows of A are perpendicular, viewed as vectors in $\mathbb{R}^n$, and any row of A has norm 1. Using this definition, prove that the following are equivalent:
   (a) Matrix A is orthogonal;
   (b) $A^tA = I_n$;
   (c) ${}^tAA = I_n$;
   (d) Any two different columns of A are perpendicular, viewed as vectors in $\mathbb{R}^n$ and any column of A has norm 1.

6. *(!)* (from Birkhoff and MacLane, "A Survey of Modern Algebra", Revised Edition, Macmillan Company)

Let A be an $(n \times n)$ strictly upper triangular matrix with real entries. This means that A has zeroes on and below the diagonal. For instance, if $n = 3$, A has the form

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}; a, b, c \in \mathbb{R}.$$

Prove that $\exists k$ a non-negative integer such that $A^{\mathrm{k}} = 0$. Hint: First try to prove this for case $n = 3$.

7. *OPTIONAL* (from Strang, "Linear Algebra and Its Applications", Second Edition, Academic Press)
Find the Jacobian determinant for the change from rectangular coordinates x, y, z to spherical coordinates r, θ, ϕ: $x = rcos\theta cos\phi$, $y = rsin\theta cos\phi$,$z = rsin\phi$.

8. (!) To an $(n \times n)$ matrix with real entries A, we associate a polynomial (called the characteristic polynomial of A) $P_A(X) = det(XI_n - A)$. Prove that for any two $(n \times n)$ matrices with real entries A, B,

$$P_{AB}(X) = P_{BA}(X).$$

Hint: First assume that A is invertible and prove the claim under this assumption. Now assume that $detA = 0$. Next consider the function $f(t) = det(tI_n + A)$. Prove that the set

$$V(f) = \{t \in \mathbb{R} \textit{ such that } f(t) = 0\}$$

is finite. Prove that

$$det(XI_n - (tI_n + A)B) = det(XI_n - B(tI_n + A))$$

holds for all $t \notin V(f)$. Prove the claim.

If you experienced difficulties in solving this you may use:
Other hints:
(A) The characteristic polynomial of a matrix is a polynomial. (Please prove this by induction. Should you have difficulties, you may use the textbook, pages 308 - 309, Fact 7.2.5.)
(B) Show that f is a polynomial of degree n in t (the proof is similar to (A)).
(C) Any (non-trivial) polynomial equation has at most finitely many solutions. (You may use this without proving it)
(D) Using (C), show that if two polynomials are equal for infinitely many values of the variable, then they are equal everywhere.
(E) Note that, once we fix X the equality

$$det(XI_n - (tI_n + A)B) = det(XI_n - B(tI_n + A)).$$

becomes an equality of polynomials in t. And now the conclusion follows easily by (D).

9. (a) Let $x \in \mathbb{R}$. Let

$$A = \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix}.$$

Compute (and provide complete justification for all your claims) A^{k} for all natural numbers k.
Hint: First solve some particular cases ($k = 2, 3, 4$). Then try to see a pattern and formulate a claim that depends on k. Now do the proof by induction on k. In your claim, the $(1, 3)$ entry of A^{k} will be $\frac{k(k-1)}{2}x^{\mathrm{k\text{-}2}}$.
(b) *OPTIONAL.* Let $x \in \mathbb{R}$. Let A be the $(n \times n)$ matrix with real entries defined as follows:

$$A = \begin{pmatrix} x & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & x & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & x & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & x & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & x & 1 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & x \end{pmatrix}.$$

For instance, if $n = 4$,

$$A = \begin{pmatrix} x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & x \end{pmatrix}$$

and if $n = 3$ it is just the matrix at part (a).
Compute A^{k} for all natural numbers k.