MAT 211, Linear Algebra, Summer 1, 2008

Homework 1 - due Wednesday, June 11

Important Note: **Please show all your work and provide full justification whenever the case arises. Please write clearly. Answers with little or no justification (when the case arises) will receive no credit. Similarly, if you do not show all your work at some problem, your answer will receive no credit. Answers that are unclearly written might not be graded.** Problems that are accompanied by an exclamation mark (!) are considered to be more challenging (or nonstandard). These problems are also mandatory. In the exams problems with higher level of difficulty or nonstandard will also be marked with an exclamation mark. Please note that in some cases, a marked nonstandard problem might be easier and shorter than the standard ones. The problems that are not explicitly written are from the official textbook.

1. Solve the linear systems in exercises 2 and 16 of section 1.1, page 5.

2. Solve the linear systems in exercises 2 and 10 of section 1.2, page 20.

3. Solve exercises 20 and 36, section 1.3, page 36.

4. An element of $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) is an $n-$uple of real numbers. We will usually view elements of $\mathbb{R}^n$ as column-vectors. For instance, if $n = 4$, an element of $\mathbb{R}^4$ is of the form $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, where x_1, x_2, x_3, $x_4 \in \mathbb{R}$. In particular, please note that with this convention, an element of $\mathbb{R}^n$ is an $(n \times 1)$ matrix with real entries (since it has n rows and 1 column). So it makes sense to multiply an $(n \times n)$ matrix A with an element (a vector) in $\mathbb{R}^n$.
   Solve exercises 48 a, b; 52 and 62 in section 1.3 pages 37, 38.

5. Let A be a (2×2) matrix with real entries. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{R}$. We define the trace of A by

   $$Tr\text{A=a+d} \in \mathbb{R}$$

   Recall that

   $$detA = ad - bc \in \mathbb{R}$$

   (a) Prove that $A^2 - (TrA)A + (detA)I_2 = 0$, where

   $$I_2=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(b) By definition, $A^0 = I_2$, $A^1 = A$, $A^2 = AA, ...$ Prove that

$$A^{n+2} - (TrA)A^{n+1} + (detA)A^n = 0 \forall n \geq 0.$$

!(c) Prove that $\forall n \geq 0, \exists \alpha, \beta \in \mathbb{R}$ such that $A^n = \alpha A + \beta I_2$. Hint: Use the following:
Principle of Mathematical Induction: Let $P(n)$ be a claim that depends on a non-negative integer n. Assume that:
(i) $P(0)$and $P(1)$ are true.
(ii) If $P(n)$ and $P(n+1)$ hold, then $P(n+2)$ holds too.
Then $P(n)$is true $\forall n \geq 0$.

6. (a) Let $a, b, c \in \mathbb{R}$. Compute $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^n$ for all $n \geq 0$.

   (b) Let A be an $(n \times n)$matrix with real entries. Let U be an $(n \times n)$ invertible matrix with real entries (this just means that $\exists U^{-1}$(another) $(n \times n)$ matrix with real entries such that $UU^{-1} = U^{-1}U = I_n$, where (as usually) $I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$. Justify the following claim: $(UAU^{-1})^{100} = UA^{100}U^{-1}$.

7. Compute the determinants of the matrices given in exercises 2,4,6,8,18,20,22,32,34,36,42 in section 6.1, page 259.

8. Do problem 56 in section 6.1, page 260. (!) In addition, compute d_n and prove your claim.

9. Let $A = (a_1, b_1)$, $B = (a_2, b_2) \in \mathbb{R}^2$. Suppose that $A \neq B$. Show that the equation of the line through A and B is given by $\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0.$

10. (after a problem in Smith, Larry, "Linear Algebra", Springer, 1998)
    If A is an $(n \times n)$ matrix with real entries, $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$, we define the transpose of A by $({}^tA)_{ij} = a_{ji} \forall i, j \in \{1, 2, ..., n\}$. For instance, if

$$A = \begin{pmatrix} 1 & -1 & 100 \\ 2 & 3 & 5 \\ 4 & 78 & 9 \end{pmatrix}, {}^tA = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 78 \\ 100 & 5 & 9 \end{pmatrix}.$$

    Recall that detA$= det^tA$.
    (a) Prove that $\forall x \in \mathbb{R}, \forall A$ an $(n \times n)$ matrix with real entries,

$$det(xA) = x^n(detA).$$

(b) Let A be an $(n \times n)$ matrix with real entries such that $A = -{}^tA$. Assume n is odd. Show that $detA = 0$.

11. Find $detA$, where $A = (a_{ij})_{1\le i\le n, 1\le j\le n}$, $a_{ij} = i + j$.
(from Strang, Gilbert, "Linear Algebra and Its Applications" , Academic Press, 1980).

EXTRA CREDIT (5%) (*Please read the extra credit policy on the course website*)

(a) Evaluate $\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$, where x_1, x_2, $x_3 \in \mathbb{R}$.

(b) Prove that

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (x_j - x_i),$$

$$where\ x_1, x_2, ..., x_n \in \mathbb{R}.$$

Hint: Use mathematical induction.

First prove the claim under the assumption that $\exists i, j \in \{1, 2, ..., n\}\ i \neq j$ such that $x_i = x_j$.

Now assume that $x_i \neq x_j \forall i \neq j$.

Consider the polynomial $P(T) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \\ 1 & T & T^2 & \cdots & T^{n-1} & T^n \end{vmatrix}$.

$P(x_1) =?$, $P(x_2) =?, ..., P(x_n) =?$ What is the degree of P?

Prove that, if the claim holds for n then it holds for $n + 1$. $P(x_{n+1}) =?$

NOTE: $\prod$= product; for instance,

$$if\ n = 3, \prod_{1\le i<j\le 3} (x_j - x_i) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2);$$

$$if\ n = 2, \prod_{1\le i<j\le 2} (x_j - x_i) = x_2 - x_1$$