MAT 211 - Introduction to Linear Algebra, Summer 1, June 23, 2008

MIDTERM

1. (a) (*3 points*) Define the concept of linear independence.
(b) (*3 points)* Give an example of a set of elements of a linear space that are NOT linearly independent. Justify your answer.

Solution.
(a) Let V be a real linear space. Let

$$\{v_1, v_2, ..., v_{\mathrm{k}}\} \sqsubseteq V.$$

$v_1, v_2, ..., v_{\mathrm{k}}$ are linearly independent if and only if

$$x_1, x_2, ..., x_k \in \mathbb{R}, x_1v_1 + x_2v_2 + ... + x_kv_k = 0_V \Rightarrow x_1 = x_2 = ... = x_k = 0,$$

where 0_V denotes the zero vector (neutral element) of V.
(b) *Example 1.* $(1, 2, 3), (5, 10, 15)$are not linearly independent because

$$(5, 10, 15) = 5(1, 2, 3).$$

Another, equally correct justification is:

$$5(1, 2, 3) + (-1)(5, 10, 15) = (0, 0, 0),$$

which shows that we have found a non-trivial liniar combination of the zero vector of $\mathbb{R}^3$ $(0, 0, 0)$ in terms of the given vectors $(1, 2, 3), (5, 10, 15)$.
Example 2. $(1, 1, 1, 1), (1, 0, 0, 0), (2, 1, 1, 1)$are not linearly independent vectors of $\mathbb{R}^4$ because

$$1(1, 1, 1, 1) + 1(1, 0, 0, 0) + (-1)(2, 1, 1, 1) = (0, 0, 0, 0),$$

and this shows that we have found a non-trivial linear combination of the zero vector in $\mathbb{R}^4$ $(0, 0, 0, 0)$ in terms of the vectors $(1, 1, 1, 1), (1, 0, 0, 0), (2, 1, 1, 1)$.
Another, equally correct justification is:

$$(2, 1, 1, 1) = (1, 1, 1, 1) + (1, 0, 0, 0).$$

Example 3. Let P_7 be the real linear space of polynomials with real coefficients of degree at most 7. More precisely,

$$P_7 = \{a_0+a_1X+a_2X^2+a_3X^3+a_4X^4+a_5X^5+a_6X^6+a_7X^7 : a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{R}\}.$$

$2X$,$X^2 + 15X^7, 3X^3 + X^5, 2X - X^2 + 3X^3 + X^5 - 15X^7$ are not linearly independent because

$$1\cdot(2X)+(-1)\cdot(X^2+15X^7)+1\cdot(3X^3+X^5)+(-1)\cdot(2X-X^2+3X^3+X^5-15X^7) = 0,$$

and hence, we have found a non-trivial linear combination of the zero vector of P_7(which is just the zero polynomial) in terms of $2X$,$X^2+15X^7, 3X^3+X^5, 2X-X^2+3X^3+X^5-15X^7$.

Example 4. Let V be any non-trivial real linear space. Let v be any non-zero element of V. Then $v, 0_V$ are not linearly independent because

$$0 \cdot v + 1 \cdot 0_V = 0_V,$$

and this proves that we have found a non-trivial linear combination of the zero vector 0_V of V in terms of v,0_V.

2. *(4 points)* Let A be a (2×2) matrix with real entries,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{R}.$$

We define the trace of A by

$$TrA = a + d \in \mathbb{R}.$$

Prove that

$$A^2 - (TrA)A + (detA)I_2 = O,$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Solution.
Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any (2×2) matrix with real entries.

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}.$$

$$(TrA)A = (a+d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix} = \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix}.$$

$$(detA)I_2 = (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}.$$

It follows that,

$$A^2 - (TrA)A + (detA)I_2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} =$$

$$= \begin{pmatrix} a^2+bc-(a^2+ad)+(ad-bc) & (ab+bd)-(ab+bd) \\ (ac+cd)-(ac+cd) & (bc+d^2)-(ad+d^2)+(ad-bc) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so we are done.

3. (a) *(3 points)* Prove that $\{(2,3,4),(5,0,1),(0,1,1)\}$ is a basis of $\mathbb{R}^3$.
(b) *(1 point)* Express the zero vector of $\mathbb{R}^3$ as a linear combination of the elements of this basis. Please note that the zero vector is the same as the neutral element.

Solution.
(a) By definition, $\{(2,3,4),(5,0,1),(0,1,1)\}$ is a basis of $\mathbb{R}^3$ if and only if the following two conditions are satisfied:
(1) $(2,3,4),(5,0,1),(0,1,1)$ are linearly independent;
(2) $\mathit{Span}(\{(2,3,4),(5,0,1),(0,1,1)\}) = \mathbb{R}^3$.
In order to prove that (1) holds, let $x_1, x_2, x_3 \in \mathbb{R}$ be such that

$$x_1(2,3,4) + x_2(5,0,1) + x_3(0,1,1) = (0,0,0).$$

This is equivalent to

$$(2x_1,3x_1,4x_1)+(5x_2,0,x_2)+(0,x_3,x_3) = (0,0,0) \Leftrightarrow (2x_1+5x_2, 3x_1+x_3, 4x_1+x_2+x_3) = (0,0,0) \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{array}{c} 2x_1+5x_2=0 \\ 3x_1+x_3=0 \\ 4x_1+x_2+x_3=0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} 2x_1+5x_2=0 \\ x_3=-3x_1 \\ 4x_1+x_2+(-3x_1)=0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} 2x_1+5x_2=0 \\ x_3=-3x_1 \\ x_1+x_2=0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} 2(-x_2)+5x_2=0 \\ x_3=3x_2 \\ x_1=-x_2 \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{c} 3x_2=0 \\ x_3=3x_2 \\ x_1=-x_2 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x_2=0 \\ x_3=0 \\ x_1=0 \end{array} \right. ,$$

so we have proved that $(2,3,4),(5,0,1),(0,1,1)$ are linearly independent.
In order to prove that (2) holds, let (a,b,c) be an arbitrary element of $\mathbb{R}^3$.We will prove that $\exists x_1, x_2, x_3 \in \mathbb{R}$ such that

$$(a,b,c) = x_1(2,3,4) + x_2(5,0,1) + x_3(0,1,1).$$

This equality of vectors in $\mathbb{R}^3$ is equivalent to

$$(a,b,c) \quad = \quad (2x_1+5x_2, 3x_1+x_3, 4x_1+x_2+x_3) \Leftrightarrow \left\{ \begin{array}{c} 2x_1+5x_2=a \\ 3x_1+x_3=b \\ 4x_1+x_2+x_3=c \end{array} \right. \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{array}{c} x_1=\frac{a-5x_2}{2} \\ x_3=b-3x_1 \\ 4x_1+x_2+b-3x_1=c \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x_1=\frac{a-5x_2}{2} \\ x_3=b-3x_1 \\ x_1+x_2=-b+c \end{array} \right. \Leftrightarrow$$

$$\Leftrightarrow \left\{ \begin{array}{c} x_1 = \frac{a-5x_2}{2} \\ x_3 = b - 3x_1 \\ x_2 = -b + c - x_1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x_1 = \frac{a-5(-b+c-x_1)}{2} \\ x_3 = b - 3x_1 \\ x_2 = -b + c - x_1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} 2x_1 = a + 5b - 5c + 5x_1 \\ x_3 = b - 3x_1 \\ x_2 = -b + c - x_1 \end{array} \right. \Leftrightarrow$$

$$\left\{ \begin{array}{c} x_1 = \frac{a+5b-5c}{-3} \\ x_3 = b + a + 5b - 5c \\ x_2 = -b + c + \frac{a+5b-5c}{3} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x_1 = \frac{a+5b-5c}{-3} \\ x_3 = a + 6b - 5c \\ x_2 = \frac{a+2b-2c}{3} \end{array} \right. .$$

This proves that, indeed

$$(a, b, c) = \frac{a + 5b - 5c}{-3}(2, 3, 4) + \frac{a + 2b - 2c}{3}(5, 0, 1) + (a + 6b - 5c)(0, 1, 1)$$

so we have proved (2) as well.
Note: This last line is obvious and one may not write it.
In order to prove condition (2) one could also argue as follows.
$dim_{\mathbb{R}}\mathbb{R}^3 = 3$. $(2, 3, 4), (5, 0, 1), (0, 1, 1)$ are linearly independent, hence they are a basis as well. Here we are using the following general theorem.
Theorem. Let V be a real linear space such that $dim_{\mathbb{R}}V = k$ and let $\{v_1, v_2, ..., v_k\}$be a system of k linearly independent elements of V. Then $\{v_1, v_2, ..., v_k\}$is a basis of V.

(b) $(0, 0, 0) = 0 \cdot (2, 3, 4) + 0 \cdot (5, 0, 1) + 0 \cdot (0, 1, 1)$.

4. *(3 points)* Compute

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^{\mathrm{k}}$$

for all natural numbers k. Provide justifications for all your claims.

Solution.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, for all $k \geq 3$,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^{k-3} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^{k-3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. *(3 points)* Compute the rank of

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

Any correct method will be accepted as long as you show all the stages of your calculations and explain clearly what theorem or definition you are using.

Solution 1.

$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ is a (3×4) matrix so clearly its rank is ≤ 3.

$$\begin{vmatrix} 6 & 0 \\ 0 & 1 \end{vmatrix} = 6$$

is a non-zero $2-$minor so clearly

$$rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \geq 2.$$

Now we will prove that

$$rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} = 2$$

by proving that any $3-$minor obtained from

$$\begin{vmatrix} 6 & 0 \\ 0 & 1 \end{vmatrix}$$

by adding a row and a column of the initial matrix is 0. There are only 2 such minors:

$$\begin{vmatrix} 2 & 3 & 0 \\ 4 & 6 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 1 & 0 & 1 \end{vmatrix}.$$

Now we compute this minors:

$$\begin{vmatrix} 2 & 3 & 0 \\ 4 & 6 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 3 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0.$$

Similarly,

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

so we are done.

Solution 2.

$rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} = dim_{\mathbb{R}} Span(\{(1,2,3,0),(2,4,6,0),(1,-1,0,1)\})$. Since $(2,4,6,0) = 2(1,2,3,0)$,it follows that

$$Span(\{(1,2,3,0),(2,4,6,0),(1,-1,0,1)\}) = Span(\{1,2,3,0),(1,-1,0,1)\}).$$

But, in a finitely generated linear space the dimension is smaller than or equal to the number of any finite set of elements that span that linear space. This shows that

$$dim_{\mathbb{R}} Span(\{(1,2,3,0),(1,-1,0,1)\}) \leq 2,$$

so

$$rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \leq 2.$$

On the other hand, $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ has a $2-$non-zero minor - for instance

$$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1,$$

so, by the definition of the rank of the matrix

$$rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \geq 2.$$

So we have proved that

$$rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} = 2.$$

We have used the following definition:
Let A be a matrix.

$$rankA = max\{k : \exists \Delta \, a\, non - zero\, k - minor\, of\, A\}.$$

Solution 3.

We do row reduction: $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 1 \end{pmatrix} \sim$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & \frac{2}{3} \\ 0 & 1 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The final form is obvously in row-reduced echelon form and the number of leading $1's$ is clearly 2, which shows that the rank of the initial matrix is 2.

Solution 4.

We know that performing "elementary operations" on the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ does not alter the rank of A because performing an elementary operation amounts to multiplying A by some invertible matrix to the left or to the right (and multiplication by invertible matrices does not change the rank).

Thus, we obtain that $rankA = rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} = rank \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 1 \end{pmatrix} =$ $\dim_{\mathbb{R}} Span\{(1,2,3,0),(0,0,0,0),(0,-3,-3,1)\} =$
$= dim_{\mathbb{R}} Span\{(1,2,3,0),(0,-3,-3,1)\} = 2$ because we can easily prove using the definition that

$$(1,2,3,0),(0,-3,-3,1)$$

are linearly independent and hence form a basis of $Span(\{(1,2,3,0),(0,-3,-3,1)\})$.

6. (!) If $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ is an $(n \times n)$ matrix with real entries we define the trace of A (denoted by TrA), by

$$TrA = \sum_{i=1}^{n} a_{ii} \in \mathbb{R}.$$

(a) *(2 points)* Prove that

$$Tr(AB) = Tr(BA)$$

for all $(n \times n)$ matrices A and B.

(b) *(2 points)* Let A be an $(n \times n)$ invertible matrix with real entries and B an $(n \times n)$ matrix with real entries such that $TrB = 5$. Compute

$$Tr(2ABA^{\text{-1}}).$$

Justify your answer.

Solution .
(a) Let $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ be any $(n \times n)$ matrices with real entries. We will denote the (r, s) entry of AB by $(AB)_{rs}$ and the (r, s) entry of BA by $(BA)_{rs}$ for all $r, s \in \{1, 2, ..., n\}$. By the definition of matrix multiplication,

$$(AB)_{rs} = \sum_{k=1}^{n} a_{rk}b_{ks}; (BA)_{rs} = \sum_{k=1}^{n} b_{rk}a_{ks} \forall r, s.$$

$$Tr(AB) = \sum_{r=1}^{n}(AB)_{rr} = \sum_{r=1}^{n}(\sum_{k=1}^{n} a_{rk}b_{kr}) =$$

$$= \sum_{k=1}^{n}(\sum_{r=1}^{n} a_{rk}b_{kr}).$$

At the last step, we just switched the two summation signs - and that is correct because the sums are finite. It follows that :

$$Tr(AB) = \sum_{k=1}^{n}(\sum_{r=1}^{n} a_{rk}b_{kr}) =^{multiplication\, of\, real\, numbers\, is\, COMMUTATIVE} \sum_{k=1}^{n}(\sum_{r=1}^{n} b_{kr}a_{rk})$$

and we obtain that:

$$Tr(AB) = \sum_{k=1}^{n}(BA)_{kk} = Tr(BA)$$

so we are done.

(b) Let $C = (c_{ij})_{i,j}$ be any $(n \times n)$ matrix with real entries and y be any real number. Then

$$Tr(yC) = \sum_{i=1}^{n}(yc_{ii}) = y\sum_{i=1}^{n} c_{ii} = y(TrC).$$

Thus, with the hypothesis given in the problem,

$$Tr(2ABA^{-1}) = 2Tr(ABA^{-1}) = 2Tr((AB)A^{-1}) =^{by\, part\, (a)} 2Tr(A^{-1}(AB)) =$$

$$= 2Tr((A^{-1}A)B) = 2Tr(I_nB) = 2TrB = 2 \cdot 5 = 10.$$

7. (!) (a) *(1 point)* Let $x \in \mathbb{R}$. Let A be an $(n \times n)$ matrix with real entries. Prove that

$$A \begin{pmatrix} x & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & x & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & x & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & x & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & x & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & x \end{pmatrix} = \begin{pmatrix} x & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & x & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & x & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & x & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & x & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & x \end{pmatrix} A.$$

(b) *(5 points)* Let B be an $(n \times n)$ matrix with real entries such that

$$BC = CB$$

holds for all $(n \times n)$ matrices with real entries C. Prove that there exists a real number a such that

$$B = \begin{pmatrix} a & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & a & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & a & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & a & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & a \end{pmatrix}.$$

In other words, you have to prove that all the entries of B that are outside the main diagonal are 0 and that all entries on the main diagonal of B are equal.

Hint: For part (b), use the canonical basis of the linear space of $(n \times n)$ *matrices with real entries, namely*

$$\{E_{\mathrm{ij}} \ \ for\ all\ i, j \in \{1, 2, ..., n\}\},$$

where the (i, j) *entry of* E_{ij} *is* 1 *and all the other are* 0.

Solution.
(a)
Let $A = (a_{ij})_{i,j}$. Direct computation shows that

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} x & 0 & ... & 0 \\ 0 & x & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & x \end{pmatrix} = \begin{pmatrix} a_{11}x & a_{12}x & ... & a_{1n}x \\ a_{21}x & a_{22}x & ... & a_{2n}x \\ ... & ... & ... & ... \\ a_{n1}x & a_{n2}x & ... & a_{nn}x \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} x & 0 & ... & 0 \\ 0 & x & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & x \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} = \begin{pmatrix} xa_{11} & xa_{12} & ... & xa_{1n} \\ xa_{21} & xa_{22} & ... & xa_{2n} \\ ... & ... & ... & ... \\ xa_{n1} & xa_{n2} & ... & xa_{nn} \end{pmatrix}.$$

Since, clearly

$$\begin{pmatrix} a_{11}x & a_{12}x & ... & a_{1n}x \\ a_{21}x & a_{22}x & ... & a_{2n}x \\ ... & ... & ... & ... \\ a_{n1}x & a_{n2}x & ... & a_{nn}x \end{pmatrix} = \begin{pmatrix} xa_{11} & xa_{12} & ... & xa_{1n} \\ xa_{21} & xa_{22} & ... & xa_{2n} \\ ... & ... & ... & ... \\ xa_{n1} & xa_{n2} & ... & xa_{nn} \end{pmatrix},$$

we have proved the claim.
(b) Since $BC = CB$ for all $(n \times n)$ matrices with real entries C,it means that, in particular,

$$BE_{\mathrm{ij}} = E_{\mathrm{ij}}B \forall i, j \in \{1, 2, ..., n\}.$$

Let $B = (b_{ij})_{i,j}$. We denote by $(E_{ij})_{rs}$the (r, s)entry of E_{ij}. By the above relation, the (r, s)entry of BE_{ij} is equal to the (r, s)entry of $E_{ij}B$ for all $r, s, i, j \in \{1, 2, ..., n\}$.

The (r, s) entry of BE_{ij} is

$$\sum_{k=1}^{n} b_{rk}(E_{ij})_{ks}$$

and the (r, s) entry of $E_{ij}B$ is

$$\sum_{k=1}^{n} (E_{ij})_{rk} b_{ks}.$$

So we obtained that

$$\sum_{k=1}^{n} b_{rk}(E_{ij})_{ks} = \sum_{k=1}^{n} (E_{ij})_{rk} b_{ks} \forall r, s, i, j.$$

Now, $(E_{ij})_{ks} = 1$if $i = k$ and $j = s$ and 0 otherwise. Thus, the left hand side of the above equality is $b_{ri}(E_{ij})_{is}$. Similarly we see that the right hand side of the above equality is $(E_{ij})_{rj}b_{js}$. So we obtained that

$$b_{ri}(E_{ij})_{is} = (E_{ij})_{rj}b_{js} \forall r, s, i, j.$$

If we take $i = r$ and $j = s$ we obtain that

$$b_{ii} = b_{jj} \forall i, j.$$

If we take $r = s = i \neq j$we obtain that

$$0 = b_{ji} \forall i \neq j,$$

so we are done.