MAT 211, Linear Algebra, Summer 1, 2008

EXTRA CREDIT READING ASSIGNMENTS

Please read the Extra Credit Policy on the course website. You may use the suggested references, other references or simply solve the problem by yourself without using any references at all.

1. (5%) Prove that an $(n \times n)$ matrix with real entries A is invertible if and only if $detA \neq 0$.
   The proof was explained in class. When studying it you may use your notes or any other references.

2. (10%) Let V be an $\mathbb{R}$-vector space such that the following hold:
   (i) $\exists\, g_1, g_2, ..., g_n \in V$ n vectors that span V. (That is, Span $\{g_1, g_2, ..., g_n\} = V$)
   (ii) $\exists f_1, f_2, ..., f_r \in V$ r linearly independent vectors of V.
   Prove that $n \geq r$. Conclude that if V has 2 basis with a finite number of elements each, then these 2 basis have the same number of elements.
   Reference: Birkhoff and MacLane, "A Survey of Modern Algebra", Revised Edition, Macmillan Company - Theorem 4 on page 168.
   NOTE: You may use (at most) 2 lines written by yourself to indicate the steps of the proof, but you should show me your 2 lines and have my permission in advance.

3. (10%) Let V be a finite dimensional $\mathbb{R}$-vector space. Let $V^* = \{\varphi : V \to \mathbb{R},\ \varphi\ linear\}$. Prove that $V^*$is an $\mathbb{R}$-vector space and dimV^*=dim$V$. Show how to construct explicitly a basis for V^* starting with a basis for V.
   Reference: Lang, "Linear Algebra", Addison-Wesley Publishing Company, 1966 - page 132, Theorem 6.

4. (10%) Let V be an $\mathbb{R}$-vector space. A linear functional on V is a linear transformation $T : V \to \mathbb{R}$. Assume that $(V, <,>)$ is a finite dimensional innner product $\mathbb{R}$-vector space. Let $T : V \to \mathbb{R}$ be a linear functional on V. Prove that there exists a unique element $A \in V$ such that

   $$T(B) =< B, A > \forall B \in V.$$

   You may assume without proof the following facts, but you should be able to state and explain their meaning at request:
   (i) If W_1 and W_2 are two finite dimensional $\mathbb{R}$-vector spaces and

   $$f : W_1 \to W_2$$

   is a linear transformation, then

   $$dimKer(f) + dimIm(f) = dimW_1.$$

(ii) If W is an $\mathbb{R}$-vector space and $\{e_1, e_2, ..., e_k\} \sqsubseteq W$ is a linearly independent system of vectors of W, then $\exists B$ a basis of W such that $\{e_1, e_2, ..., e_k\} \sqsubseteq B$.
(iii) Any $\mathbb{R}$-vector space has an orthonormal basis.
Reference: Smith Larry, "Linear Algebra", Third Edition, Springer - Theorem 15.4.5 on pages 294, 295.
NOTE: You may use (at most) 2 lines written by yourself to indicate the steps of the proof, but you should show me your 2 lines and have my permission in advance.